Mathematics Qualifying Exam Subjects

Contents

Algebra

Main Reference

- Serge Lang - *Algebra* (Sections 1.2-1.8, 2.1-2.5, 3.1-3.5, 3.7, 4.1-4.3, 5.1-5.5)

Additional References

- David S. Dummit and Richard M. Foote - *Abstract Algebra*
- Thomas W. Hungerford - *Algebra*

Topics

- **Groups:** Groups, subgroups, quotient groups, Lagrange's theorem, cyclic groups, homomorphisms, isomorphism theorems, symmetric, alternating and dihedral groups, direct products, free abelian groups, finitely generated abelian groups, group actions, Sylow subgroups.
- **Rings:** Rings, subrings, homomorphisms, ideals, prime and maximal ideals, quotient rings, isomorphism theorems for rings, direct products and Chinese remainder theorem, ring of quotients and localization, unique factorization domains, Euclidean domains, polynomial rings, factorization in polynomial rings (irreducibility).
- **Fields:** Field extensions: algebraic and transcendental extensions, simple extensions and their characterization, finite fields.
- **Modules:** Direct sum of Modules, Free Modules, Vector Spaces, Modules over Principal Ideal Domains.

Background

No specific background listed.

Analysis

Main Reference

- G. Folland - *Real Analysis*

Additional References

- Robert G. Bartle - *Elements of Integration Theory*
- E. Kreyszig - *Introductory Functional Analysis and Applications*

Topics

- **Measure Theory:** Abstract measure spaces, Lebesgue integral, convergence theorems, product measures Fubini-Tonelli, change of variables, decomposition of measures, Radon-Nikodym, L^p-spaces
- **Functional Analysis:** Normed Banach spaces, Hilbert space theory, Baire category theorem, uniform boundedness principle, open mapping theorem, closed graph theorem, Hahn-Banach theorem

Background

Calculus and undergraduate real analysis.

Algebraic Number Theory

Main Reference

- Daniel Marcus - *Number Fields* (Chapters 1-5)

Additional References

- Gerald J. Janusz - *Algebraic Number Fields*

Topics

Number Fields, Ring of integers, Discriminant, norm, trace, prime decomposition in ring of integers, Dedekind Domains, Inertia and Decomposition groups, ideal class group, splittings of prime ideals, cyclotomic fields.

Background

Field extensions and Galois Theory is necessary.

Graph Theory

Main Reference

- D. B. West - *Introduction to Graph Theory* (Prentice Hall)

Additional References

- R. J. Wilson - *Introduction to Graph Theory* (Pearson)
- J. A. Bondy and U. S. R. Murty - *Graph Theory with Applications* (North-Holland)

Topics

Fundamental concepts in graph theory, trees and distance: spanning trees, optimization, paths, cycles, Eulerian graphs, planar graphs, graph parameters, matching and factors: matching in bipartite graphs, matching in general graphs, graph algorithms, graph colorings: vertex coloring, edge coloring, Hamilton cycles, connectivity.

Background

No specific background listed.

Combinatorial Design Theory

Main Reference

- C. C. Lindner and C. A. Rodger - *Design Theory* (CRC Press)

Additional References

- W. D. Wallis - *Introduction to Combinatorial Designs* (Chapman & Hall, CRC)

Topics

Steiner triple systems, λ-fold triple systems, block designs, t-designs, intersection of designs, graph decompositions, Latin squares, mutually orthogonal Latin squares, difference sets, embeddings of combinatorial structures, Hadamard matrices.

Background

No specific background listed.

Algebraic Geometry

Main Reference

- Hartshorne - *Algebraic Geometry* (Chapter 1, Chapter 2 (Sections 1, 2, 3))

Additional References

- Mumford - *Red book of varieties and schemes*
- Fulton - *Intersection Theory*

Topics

Affine and projective varieties, morphisms, rational maps, non-singular varieties, non-singular curves, intersection theory in projective space, Bezout's theorem, sheaves, definition of schemes, affine and projective schemes.

Background

Knowledge of commutative algebra on the level of Atiyah-MacDonald's *Introduction to Commutative Algebra* is necessary.

Topological Data Analysis

1. Prerequisites / Background Knowledge

Students should have a solid graduate-level background in:

- **Algebraic Topology**: simplicial complexes, singular homology, cohomology, relative homology, Mayer–Vietoris sequence, homotopy invariance, CW complexes.
  Suggested source: A. Hatcher, *Algebraic Topology*, Ch. 0–3.
- **Linear Algebra**: vector spaces, inner products, rank–nullity, singular value decomposition.
- **Basic Probability and Statistics**: random variables, distributions, expectations, empirical measures.
- **Metric Geometry**: metric spaces, Lipschitz maps, Hausdorff distance, basics of Gromov–Hausdorff distance.

2. Core Topics in TDA

A. Simplicial and Cell Complexes in Data Analysis

- Point cloud data and metric representations.
- Simplicial complexes: Vietoris–Rips, Čech, witness complexes.
- Nerve Theorem (statement and proof in the finite case).
- Filtrations of complexes.

Sources: H. Edelsbrunner & J. Harer, *Computational Topology: An Introduction* (Ch. 3–4).

G. Carlsson, *Topology and Data*, Bull. AMS, 46(2), 2009.

B. Persistent Homology

- Definition of persistent homology and persistence modules.
- Stability theorem for persistent homology.
- Reduction algorithms for boundary matrices.
- Persistence diagrams and barcodes.
- Extended persistence.

Sources: H. Edelsbrunner & J. Harer, *Computational Topology: An Introduction* (Ch. 6–7).

F. Chazal et al., *The Structure and Stability of Persistence Modules*, SpringerBriefs, 2016.

C. Stability and Metrics

- Bottleneck distance and Wasserstein distance between persistence diagrams.
- Stability theorem of Cohen–Steiner, Edelsbrunner, Harer.

- Interleaving distance for persistence modules.

Sources: D. Cohen-Steiner, H. Edelsbrunner, J. Harer, *Stability of Persistence Diagrams*, Discrete Comput. Geom., 37(1), 2007.

F. Chazal et al., *An Introduction to Topological Data Analysis: Fundamental and Practical Aspects*, 2021.

D. Mapper and Reeb Graphs

- Definition of the Reeb graph and Mapper construction.
- Stability of Reeb graphs.
- Applications to shape analysis.

Sources: G. Singh, F. Mémoli, G. Carlsson, *Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition*, SPBG, 2007.

M. de Silva, V. de Silva, *Categorified Reeb Graphs*, Discrete Comput. Geom., 2021.

3. Expected Knowledge for the Exam

Students should be able to:

- State and explain main definitions and theorems (including stability theorems).
- Construct simple Vietoris–Rips and Čech complexes from small point sets.
- Compute persistent homology by hand for simple examples.
- Explain the algorithmic pipeline from point cloud to persistence diagram.
- Apply the Nerve Theorem in the context of Čech complexes.
- Compare and contrast different filtrations.
- Outline proofs of key results (e.g., stability of persistence diagrams).

Statistics

Main Reference

- Hogg, R. V., McKean, J., Craig, A. T. (2019) - *Introduction to Mathematical Statistics* (8th ed.). Pearson. (Sections 3.3-3.6, 6.1-6.5, 7.1-7.9, 8.1-8.3)

Additional References

- Casella, G., Berger, R. L. (2002) - *Statistical Inference* (2nd ed.). Duxbury Press.
- Bain, L. J., Engelhardt, M. (1994) - *Introduction to Probability and Mathematical Statistics* (2nd ed.). Duxbury Press.

Topics

- **Statistics and Sampling Distributions:** Normal Distribution, t-Distribution, F-Distribution, Chi-Square Distribution.
- **Methods of Estimation:** Maximum Likelihood Estimation (MLE), Method of Moments (MoM).
- **Properties of Estimators:** Unbiasedness, Efficiency, Mean Squared Error (MSE), Consistency, Asymptotic Normality, Asymptotic Efficiency.
- **Sufficient Statistics, Completeness, and Exponential Family:** Neyman-Factorization Theorem, Basu's Theorem, Exponential Family Representation.
- **Large-Sample Properties of Estimators:** Fisher Information, Cramér-Rao Lower Bound, Likelihood Expansions, Score Function.
- **Interval Estimation:** Pivotal Quantities, Confidence Intervals (Exact and Asymptotic).
- **Hypothesis Testing:** Neyman-Pearson Lemma, Uniformly Most Powerful (UMP) Tests, Generalized Likelihood Ratio Tests.

Background

No specific background listed.

Probability Theory

Main Reference

- Erhan Cinlar - *Probability and Stochastics* (Chapters 1-6)
- Ioannis Karatzas and Steven E. Shreve - *Brownian Motion and Stochastic Calculus* (Chapters 1-5)

Additional References

- Fima C. Klebaner - *Introduction to Stochastic Calculus with Applications* (Chapters 2-7)

Topics

Probability spaces, convergence, conditioning, martingales, stopping times, Brownian motion, stochastic integration, Fokker-Planck equation, stochastic differential equations, diffusion processes.

Background

Real Analysis I and II.

Analytic Number Theory

Main Reference

- T. M. Apostol - *Introduction to Analytic Number Theory* (all chapters)

Additional References

- M. R. Murty - *Problems in Analytic Number Theory*

Topics

Arithmetic functions from an algebraic and analytic point of view, Dirichlet series, primes in arithmetic progressions, Dirichlet's theorem (both analytic and elementary proofs), Prime Number Theorem (with error terms), Contour integration methods in the theory of primes, Functional equations (both for the Riemann zeta function and Dirichlet L-functions), Gauss sums, Gamma function, theory of infinite (Hadamard) products and entire functions, zero-free regions, explicit formulas for the count of primes in terms of zeros of zeta functions, Phragmen-Lindelöf theorem, Selberg class of zeta functions, sieve methods (especially Brun's sieve, Selberg's sieve, Linear sieve), p-adic methods and p-adic versions of the above theories.

Background

Real Analysis, Complex Analysis, some algebra (at least a knowledge of basics of groups, rings and fields).

Symplectic Geometry

Main Reference

- Cannas da Silva - *Lectures on symplectic geometry*

Additional References

- McDuff, Salamon - *Introduction to symplectic topology*

Topics

Symplectic Linear Algebra, Symplectic Manifolds, Lagrangian Submanifolds, Darboux's Theorem and Moser's Technique, Cotangent Bundles and Generating Functions, Hamiltonian Mechanics, Moment Maps, Hamiltonian Group Actions, Symplectic Reduction, Almost Complex Structures, Kähler Manifolds, Contact Manifolds, Symplectic Toric Manifolds, Convexity Theorems, The Duistermaat-Heckman Theorem.

Background

Basic smooth manifold theory, including some familiarity with Lie groups and differential forms.

Riemannian Geometry

Main Reference

- M. P. do Carmo - *Riemannian Geometry* (Chapters 1-7)

Additional References

- P. Petersen - *Riemannian Geometry*

Topics

Differentiable manifolds, Riemannian metrics, affine connections, Levi-Civita connection, parallel transport, geodesics, curvature, Riemann curvature tensor, sectional curvature, Ricci curvature, scalar curvature, first and second variations of arc length, Jacobi fields, conjugate points, Hopf-Rinow theorem, Hadamard-Cartan theorem, isometric immersions, second fundamental form.

Background

No specific background listed.

Numerical Analysis

Main References

- Holger Wendland – *Numerical Linear Algebra: An Introduction* (Chapters 2-5, Sections 6.1-6.2)
- Endre Süli and David F. Mayers – *An Introduction to Numerical Analysis* (Chapters 1, 4, 6-8, Sections 9.1-9.4, 10.1-10.5)
- Arieh Iserles – *A First Course in the Numerical Analysis of Differential Equations (2nd Edition)* (Chapters 1-4, 8, Sections 16.1-16.5)

Additional References

- Lloyd N. Trefethen and David Bau – *Numerical Linear Algebra*
- Uri M. Ascher and Chen Greif – *A First Course in Numerical Methods*

Topics

- **Numerical Linear Algebra:** Matrix factorizations including QR factorization and singular value decomposition, stability and conditioning, iterative and direct solutions of linear systems and least-squares problems, calculation of eigenvalues and singular values, Krylov subspace methods.
- **Approximation Theory and Nonlinear Equations:** Solutions of a nonlinear equation and systems of nonlinear equations, Newton's method, polynomial interpolation, best polynomial approximations in the 2-norm and in the ∞-norm, orthogonal polynomials, numerical integration, Gaussian quadrature.
- **Numerical Differential Equations:** One-step, multistep and Runge-Kutta methods for the initial value problem, stiffness and A-stability, finite-difference schemes for the Poisson and diffusion equations.

Background

Advanced calculus and linear algebra.

Optimization

Main References

- Jorge Nocedal and Stephen J. Wright – *Numerical Optimization (2nd Edition)* (Chapters 2-3, 6, 10, 12, Sections 17.1-17.2)
- Stephen Boyd and Lieven Vandenberghe – *Convex Optimization* (Chapters 2-5, Sections 9.1-9.5, 11.1-11.7)

Additional References

- Jonathan M. Borwein and Adrian S. Lewis – *Convex Analysis and Nonlinear Optimization*

Topics

Line search methods, Newton and quasi-Newton methods for unconstrained optimization, nonlinear least-squares problems, nonlinear equations, optimality conditions for constrained optimization, basics of convex optimization, duality for convex optimization, interior-point methods.

Background

Advanced calculus and linear algebra.

Complex analysis

Main References

- E. Freitag, R. Busam – *Complex Analysis*

Additional References

- L. Ahlfors – *Complex Analysis*

Topics

Complex numbers, complex derivative, Cauchy-Riemann equations, Cauchy's integral theorem & formula, power series, singularities of analytic functions, Laurent series, residue theorem

Background

Calculus and undergraduate real analysis