Hypocoercivity, hypocontractivity and short-time decay of solutions of linear evolution equations

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Outline

Semi-dissipative evolution eq.

Consider a linear evolution equation:

$$\dot{x} = Ax = -Bx = (J - R)x,$$

J skew-adjoint, R self-adjoint.

- ▷ $A = -B$ linear operator acting on Hilbert space $\mathcal{H}$.
- ▷ $A = -B$ semi-dissipative (R positive semidefinite) and a unique steady state $Ax_\infty = 0$ exists.

Motivating questions:

1. Optimal long-time decay estimate.
   - Exponential decay: $\|x(t) - x_\infty\| \leq ce^{-\mu t}\|x(0) - x_\infty\|$, $t > 0$.
   - (sharp) maximal rate $\mu > 0$ and minimal $c \geq 1$ (uniform in $x(0)$).
2. Short-time decay estimate.
   - $\|x(t) - x_\infty\| \leq (1 - ct^\alpha + \mathcal{O}(t^{\alpha+1}))\|x(0) - x_\infty\|$, $t \in [0, \epsilon)$.
   - Relation to spectral properties of operator $A = -B$.

Definition

Consider evolution equation $\dot{x} = Ax = (-B)x = (J - R)x$. B is coercive if $\langle x, Bx \rangle \geq \kappa \|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$ and some $\kappa > 0$.

Classical: If B is coercive then solution exponentially decays (is asymptotically stable).
But this is not necessary:

Example: $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \in \mathbb{C}^{2,2}$, Evs $\frac{1}{2}(-1 \pm \sqrt{3}i)$, decay rate $\frac{1}{2}$.
However, $B = -A = -(J - R)$ is not coercive, since R is only semidefinite.

Analysis via Hypocoercivity

- Notion hypocoercivity was introduced for $\dot{x} = -Bx$, $x(0) = x_0$ on Hilbert space $\mathcal{H}$, where linear operator B is not coercive, but solutions still exhibit exponential decay in time.
- More precisely, for hypocoercive operators $A = -B$, there exist constants $\lambda > 0$ and $c \geq 1$, such that

$$\|e^{-Bt}x_0\|_{\widetilde{\mathcal{H}}} \leq c\, e^{-\lambda t}\|x_0\|_{\widetilde{\mathcal{H}}} \qquad \text{for all } x_0 \in \widetilde{\mathcal{H}}\,, \qquad t \geq 0\,,$$

where $\widetilde{\mathcal{H}}$ is Hilbert space, densely embedded in $(\ker B)^\perp \subset \mathcal{H}$.

- C. Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202, 2009.

Outline

Stability analysis for linear ODEs

Classical characterization via spectrum of $A \in \mathbb{C}^{n,n}$..

Theorem

Consider a constant coefficient system $\dot{x} = Ax$ with $A \in \mathbb{C}^{n,n}$.

- *It is asymptotically stable if all eigenvalues of A have negative real part.*
- *It is stable if all evs of A have nonpositive real part and all evs with real part 0 are semisimple.*

However, Jordan structure is hard to compute numerically.

One may use pseudospectra or employ semi-dissipativity.

- L.N. Trefethen and M. Embree. Spectra and Pseudospectra. Princeton University Press, 2020.

Lyapunov characterization of stability

- The continuous-time system is stable if there exists a solution $P > 0$ of the Lyapunov inequality

$$A_d^H P + P A_d \leq 0.$$

- The continuous-time system is *asymptotically stable* if there exists a solution $P > 0$ of the strict Lyapunov inequality

$$A_d^H P + P A_d < 0.$$

Open question: Which solution of the Lyapunov inequality. Optimize a robustness measure?

Lemma (Achleitner, Arnold, M. 2021)

Let $J, R \in \mathbb{C}^{n,n}$ with $J^H = -J$ and $R^H = R \geq 0$. T.f.a.e.

1. *There exists integer $m \geq 0$ such that $\operatorname{rank}[R, JR, \ldots, J^m R] = n$.*
2. *There exists integer $m \geq 0$ such that $\mathcal{T}_m := \sum_{j=0}^{m} J^j R (J^H)^j > 0$.*
3. *No eigenvector of J lies in the kernel of R.*
4. *$\operatorname{rank}[\lambda I - J, R] = n$ for every $\lambda \in \mathbb{C}$.*

Moreover, the smallest possible m in 1. and 2. coincide.

▷ F. Achleitner, A. Arnold, and V. Mehrmann. Hypocoercivity and controllability in linear semi-dissipative ODEs and DAEs. Vol. 103, *ZAMM, Zeitschrift f. Angewandte Mathematik und Mechanik*, e202100171, 2021. http://arxiv.org/abs/2104.07619 https://doi.org/10.1002/zamm.202100171

Definition

Let $J, R \in \mathbb{C}^{n,n}$ with $J^H = -J$ and $R^H = R \geq 0$.
The hypocoercivity index (HC-index) $m_{HC}(A)$ of $A = J - R$ is the smallest integer m such that $\sum_{j=0}^{m} J^j R (J^H)^j > 0$. If m is finite then we call A hypocoercive.

Is there a numerically feasible way to check hypocoercivity?

Lemma

Let $J = -J^H$, $R = R^H \in \mathbb{C}^{n,n}$. There exists unitary P, such that

$$PJP^H = \left[\begin{array}{cccccc|c} J_{1,1} & -J_{2,1}^H & & \cdots & & 0 & 0 \\ J_{2,1} & J_{2,2} & -J_{3,2}^H & & & & \\ & \ddots & \ddots & \ddots & & \vdots & \\ & & J_{k,k-1} & J_{k,k} & -J_{k+1,k}^H & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \\ & & & & J_{s-2,s-3} & J_{s-2,s-2} & -J_{s-1,s-2}^H & \\ 0 & & \cdots & & & J_{s-1,s-2} & J_{s-1,s-1} & 0 \\ \hline 0 & & & \cdots & & & 0 & J_{ss} \end{array}\right], \quad PRP^H = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

w. block sizes $n_1 \geq \cdots \geq n_{s-1} \geq n_s \geq 0$, $n_{s-1} > 0$, $R_1 \in \mathbb{C}^{n_1,n_1}$ nonsingular. If R is nonsingular, then $s = 2$ and $n_2 = 0$. If R is singular, then $s \geq 3$ and the matrices $J_{i,i-1}$, $i = 2, \ldots, s-1$, in the subdiagonal have full row rank and are of the form

$$J_{i,i-1} = \begin{bmatrix} \Sigma_{i,i-1} & 0 \end{bmatrix}, \quad i = 2, \ldots, s-1,$$

with nonsingular matrices $\Sigma_{i,i-1} \in \mathbb{C}^{n_i,n_i}$.

Lemma

Let $A = J - R$ be a semi-dissipative matrix transformed to staircase form. Then the matrix A is hypocoercive if and only if $n_s = 0$, i.e., the last row and last column in PJP^H are absent, and the HC-index of A_c is $m_{HC} = s - 2$.

We can check hypocoercivity via staircase form and compute HC-index in a numerically stable way.
BUT we need rank decisions in finite precision.
Perturbation theory for staircase form open problem.
Best to use structure of operator.

Relation between concepts

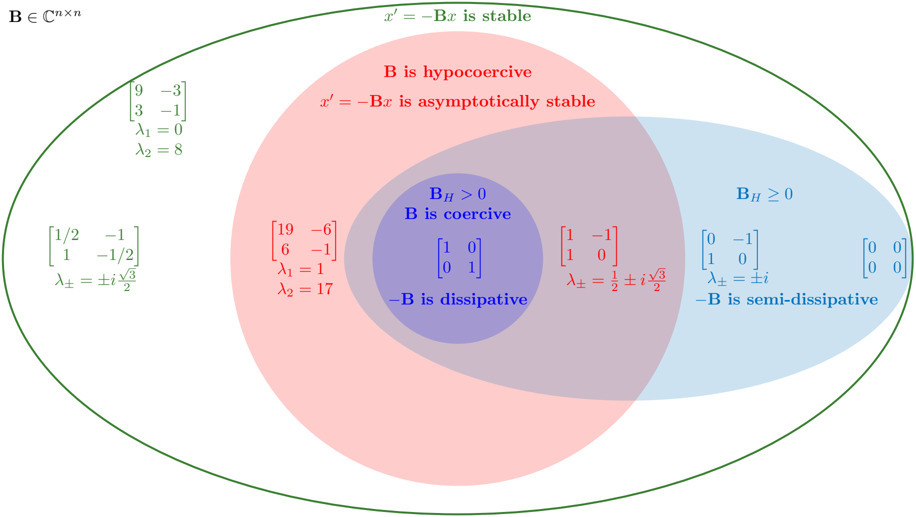

Figure: $-B = A = J - R$, $B_H = R$: (hypo)coercive (pink), $R \geq 0$ (blue), and for which solutions of $\dot{x} = Ax$ are stable (white).

Relation between short-time decay of $\|e^{At}\|_2$ and HC-index.

Theorem

Consider a semi-dissipative Hamiltonian ODE whose system matrix A has finite HC-index. Its (finite) HC-index is m_{HC} if and only if

$$\|e^{At}\|_2 = 1 - ct^a + \mathcal{O}(t^{a+1}) \quad \text{for } t \in [0, \epsilon),$$

where $c > 0$ and $a = 2m_{HC}(A) + 1$.

Open problem what happens between ϵ and the point when asymptotic behaviour sets it?

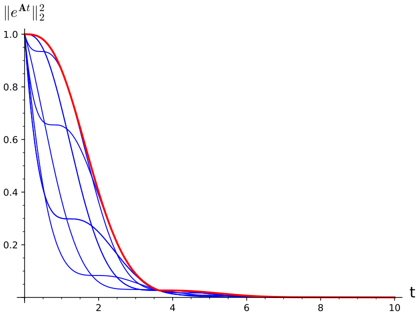

Figure: For the ODE with $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, the squared propagator norm ($\|e^{At}\|_2^2 \sim 1 - t^3/6 + \mathcal{O}(t^4)$ for $t \to 0+$), (red line) and the squared norms of a family of solutions (blue lines) are plotted. The squared propagator norm is not continuously differentiable at $t = 2\pi/\sqrt{3}$, it is the envelope of $\|x(t)\|_2^2$ for all solutions with $\|x(0)\|_2^2 = 1$.

Stability of discrete time systems.
Classical characterization via Jordan canonical form of $A \in \mathbb{C}^{n,n}$.

Theorem

Consider a constant coefficient system
$x_{k+1} = A_d x_k \,, \qquad k = 0, 1, 2, \ldots$ *with* $A_d \in \mathbb{C}^{n,n}$.

- *It is asymptotically stable if all eigenvalues of* A_d *are in the open unit disk.*
- *It is stable if all evs of* A_d *are in the closed unit disk and all evs on unit circle are semisimple.*

However, Jordan structure is hard to compute numerically.

One may use pseudospectra or employ semi-dissipativity.

- L.N. Trefethen and M. Embree. Spectra and Pseudospectra. Princeton University Press, 2020.

Lyapunov characterization of stability

- The discrete-time system is stable if there exists a solution $P > 0$ of the discrete Lyapunov (Stein) inequality

$$A_d^H P A_d - P \leq 0.$$

- The discrete-time systerm is *asymptotically stable* if there exists a solution $P > 0$ of the discrete Lyapunov (Stein) inequality

$$A_d^H P A_d - P < 0.$$

Open question: Which solution of the Lyapunov inequality. Optimize a robustness measure?

Definition

Let $\sigma_{\max}(A_d)$ be the largest singular value (the *spectral norm*) of A_d. We call A_d *contractive* if $\sigma_{\max}(A_d) < 1$; and we call A_d *semi-contractive* if $\sigma_{\max}(A_d) \leq 1$.
A matrix A_d is called *hypocontractive* if all eigenvalues of A_d have modulus strictly less than one.

Consequently, a discrete-time system is asymptotically stable if and only if the system matrix A_d is hypocontractive.

Hypocontractivity and controllability.

Lemma

Let $A_d \in \mathbb{C}^{n,n}$ be semi-contractive. T.f.a.e.

- *There exists an integer $m \geq 0$ such that* $\operatorname{rank}[(I - A_d^H A_d), A_d^H(I - A_d^H A_d), \ldots, (A_d^H)^m(I - A_d^H A_d)] = n$.
- *There exists an integer $m \geq 0$ such that* $D_m := \sum_{j=0}^{m} (A_d^H)^j (I - A_d^H A_d) A_d^j > 0$.
- *No eigenvector of A_d lies in the kernel of $(I - A_d^H A_d)$.*
- $\operatorname{rank}[\lambda I - A_d^H, I - A_d^H A_d] = n$ *for every $\lambda \in \mathbb{C}$, in particular for every eigenvalue λ of A_d.*

Moreover, the smallest m in first two conditions coincide.
This is controllability of the pair $(A_d^H, I - A_d^H A_d)$.
Similar result (in different notation) via observability, O. Staffans, Well-posed linear systems, Cambridge Univ. Press, 2005.

Hypocontractivity index

Definition

For a semi-contractive matrix A_d, we define the hypocontractivity index or discrete HC-index (dHC-index) m_{dHC} as the smallest integer (if it exists) such that the second condition in the Lemma holds. For semi-contractive matrices A_d that are not hypocontractive we set $m_{dHC} = \infty$.

A semi-contractive matrix A_d is contractive iff $m_{dHC} = 0$.
In operator theory $(I - A_d^H A_d)^{\frac{1}{2}}$ is called the *defect operator* of A_d and the closure of its image the *defect space* with its dimension being called the *defect index*. The defect operator and its index are a measure for the distance of an operator from being unitary.

Theorem

Let A_d be semi-contractive with finite hypocontractivity index. Its (finite) hypocontractivity index is m_{dHC} if and only if

$$\|A_d^j\|_2 = 1 \text{ for all } j = 1, \dots, m_{dHC} \text{ , and } \|A_d^{m_{dHC}+1}\|_2 < 1 \ .$$

Polar decomposition is the discrete-time analogue of the additive splitting of a matrix into its Hermitian and skew-Hermitian part.

Lemma (Polar decomposition)

Let $A_d \in \mathbb{C}^{n,n}$.

(a) *There exist positive semi-definite Hermitian matrices* P_d, Q_d *and a unitary matrix* U_d *such that*

$$A_d = P_d U_d = U_d Q_d.$$

The factors P_d, Q_d *are uniquely determined and if* A_d *is nonsingular, then* $U_d = P_d^{-1} A_d = A_d Q_d^{-1}$ *is unique..*

(b) *If* A_d *is real, then* P_d, Q_d *and* U_d *may be taken to be real.*

A_d with polar decomp. $A_d = P_d U_d = U_d Q_d$ is semi-contractive iff spectra of P_d or Q_d are contained in $[0, 1]$.

Hypocontractivity via polar factors

Lemma

Let U be a unitary matrix, and H be a semi-contractive Hermitian matrix. T.f.a.e.

- *There exists an integer $m \geq 0$ such that*

$$\operatorname{rank}[(I-H), U^H(I-H), \ldots, (U^H)^m(I-H)] = n\,.$$

- *There exists an integer $m \geq 0$ such that*

$$\hat{D}_m := \sum_{j=0}^{m} (U^H)^j (I-H) U^j > 0\,.$$

- *No eigenvector of U lies in the kernel of $I-H$.*
- $\operatorname{rank}[\lambda I - U^H, I-H] = n$ *for every $\lambda \in \mathbb{C}$, in particular for every eigenvalue λ of U^H.*

Lemma (Staircase form for (U, H))

Let U be a unitary matrix, and H be a nonzero semi-contractive Hermitian matrix. Then there exists a unitary matrix P, such that PHP^H and PUP^H are block upper Hessenberg matrices of the form

$$PUP^H = \left[\begin{array}{ccccc|c} U_{1,1} & U_{1,2} & \cdots & \cdots & U_{1,s-1} & 0 \\ U_{2,1} & U_{2,2} & U_{2,3} & \cdots & U_{2,s-1} & 0 \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & U_{s-2,s-3} & U_{s-2,s-2} & U_{s-2,s-1} & 0 \\ 0 & \cdots & 0 & U_{s-1,s-2} & U_{s-1,s-1} & 0 \\ \hline 0 & \cdots & & & 0 & U_{s,s} \end{array}\right] \begin{array}{c} n_1 \\ n_2 \\ \vdots \\ n_{s-2} \\ n_{s-1} \\ n_s \end{array},$$

$$P\,H\,P^H = \left[\begin{array}{ccccc|c} H_1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I_{n_2} & 0 & \cdots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & I_{n_{s-1}} & 0 \\ \hline 0 & 0 & \cdots & \cdots & 0 & I_{n_s} \end{array}\right],$$

where $n_1 \geq n_2 \geq \cdots \geq n_{s-1} \geq n_s \geq 0$, $n_{s-1} > 0$, and $H_1 = H_1^H \in \mathbb{C}^{n_1,n_1}$ is contractive. If H is contractive, then $s = 2$ and $n_2 = 0$. If H is not contractive, then $s \geq 3$, $U_{i,i-1}$, $i = 2, \ldots, s-1$, have full row rank and are of form $U_{i,i-1} = \begin{bmatrix}\Sigma_{i,i-1} & 0\end{bmatrix}$, $i = 2, \ldots, s-1$, with nonsingular matrices $\Sigma_{i,i-1} \in \mathbb{C}^{n_i,n_i}$.

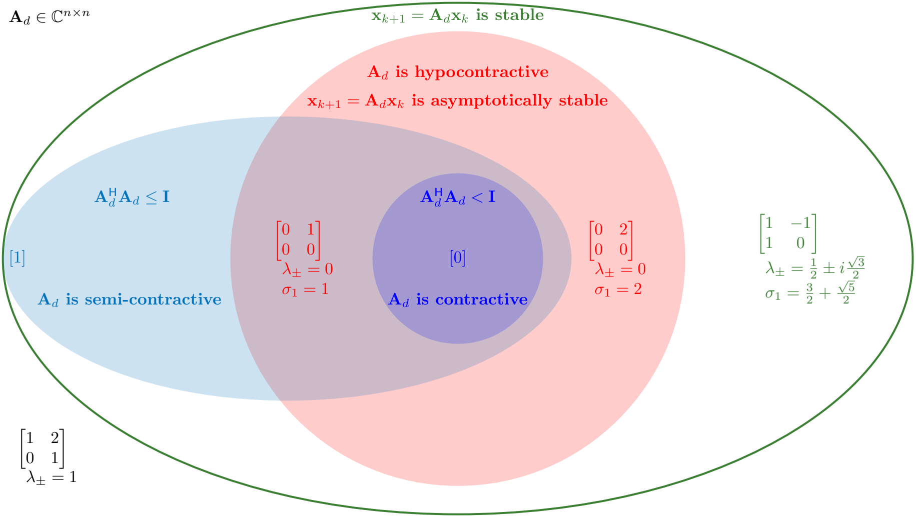

Figure: Relation between matrices A_d which are (semi-)contractive, hypocontractive and those for which the discrete-time system

Linear differential-algebraic equation (DAE)
$E\dot{x} = Ax = (J - R)x$, $E, A \in \mathbb{C}^{n,n}$, J skew-adjoint, E, R self-adjoint.
The DAE is called semidissipative (or dissipative Hamiltonian) if E, R are positive semidefinite.

Theorem

Consider system $E\dot{x} = Ax = (J - R)x$ with $E, A \in \mathbb{C}^{n,n}$

- *It is asymptotically stable if the pair (E, A) is regular $(\det(\lambda E - A) \neq 0)$, and all evs of $\lambda E - A$ have neg. real part.*
- *It is stable if (E, A) is regular, all evs of A have non-positive real part and all evs with real part 0 are semisimple.*

Pseudospectra

- M. Embree and B. Keeler. Pseudospectra of matrix pencils for transient analysis of differential-algebraic equations. SIAM J. Matrix Analysis and Applications 38, 1028-1054, 2017.

Theorem (Mehl/M./Wojtylak 2018)

Let $E \in \mathbb{C}^{n,n}$ *and* $J = -J^H$, $R = R^H \in \mathbb{C}^{n,n}$ *be such that* $R \geq 0$, $E^H = E \geq 0$. *Then the following holds for* $P(\lambda) = \lambda E - (J - R)$.

1. *If* $\mu \in \mathbb{C}$ *is an eigenvalue of* $P(\lambda)$ *then* $\mathrm{Re}(\mu) \leq 0$*;*
2. *If* $\omega \in \mathbb{R}$ *and* $\mu = i\omega$ *is an eigenvalue of* $P(\lambda)$ *then* μ *is semisimple. Moreover, if the columns of* $V \in \mathbb{C}^{n,k}$ *form a basis of the deflating subspace associated with* μ *of* $\lambda E - J$ *, then* $RV = 0$.
3. *Kronecker blocks at* ∞ *are at most of size two.*
4. *The pencil* $\lambda E - (J - R)$ *is singular iff the kernels of the matrices* E, J*, and* R *have a nontrivial intersection.*

- Mehl, V. M., and Wojtylak, *Linear algebra properties of dissipative Hamiltonian descriptor systems*. SIAM Journal Matrix Analysis and Applications, Vol. 39, 489–1519, 2018.
- Mehl, V.M., Wojtylak. *Distance problems for dissipative Hamiltonian systems and related matrix polynomials* Linear Algebra and its Application, 2021.

Lemma

Consider a semi-dissipative DAE. Then there exist nonsingular matrices L, Z such that

$$\hat{E} := L\,E\,Z =: \begin{bmatrix} E_{1,1} & 0 & 0 & 0 & 0 \\ 0 & E_{2,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{A} := L\,A\,Z =: \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & A_{2,2} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The two matrices are partitioned in the same way, with (square) diagonal block matrices of sizes $n_1, n_2, n_3, n_4 = n_1, n_5$. If the matrices $E_{1,1}$ and $E_{2,2}$ are present, then they are Hermitian positive definite.
The Hermitian part of $A_{2,2}$ is positive semidefinite.

Definition

Consider a linear DAE system with a regular pencil $\lambda E - (J - R)$ and the unitarily congruent DAE in staircase form. If the underlying implicit ODE is present then the *HC-index m_{HC}* of $\lambda E - (J - R)$ is defined as the HC-index of the system matrix $(E_{2,2}^{1/2})^{-1} A_{2,2} (E_{2,2}^{1/2})$, otherwise it is defined as 0. If the HC-index is finite then the DAE is called hypocoercive.

Consider semi-norm $\|x(t)\|_E = \langle x, Ex\rangle^{\frac{1}{2}}$ and let $S(t)$ be the evolution operator of the DAE $E\dot{x} = Ax = (J - R)x$ with $E, A \in \mathbb{C}^{n,n}$, J skew-adjoint, E, R self-adjoint, semi-definite.

Theorem

Consider a semi-dissipative DAE with a regular, hypocoercive pencil $\lambda E - (J - R)$, non-trivial dynamics, and consistent initial condition $x(0)$. Then its (finite) HC-index is m_{HC}, if and only if

$$\|S(t)\|_E = 1 - ct^a + \mathcal{O}(t^{a+1}) \qquad \text{for } t \in [0, \epsilon),$$

where $c > 0$ and $a = 2m_{HC} + 1$, and the semi-norm is

$$\|S(t)\|_E := \sup_{\|x(0)\|_E = 1, \text{for consistent } x(0)} \|x(t)\|_E, \qquad t \geq 0.$$

For $\varepsilon > 0$, consider linear semi-dissipative DAE system with

$$E = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; J = \begin{bmatrix} 0 & 1 & 1 & \varepsilon \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ -\varepsilon & 0 & 0 & 0 \end{bmatrix}, \; R := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in almost Kronecker form with $n_1 = n_2 = n_3 = n_4 = 1$.
For given $y_2(0) \in \mathbb{R}$, the solution is

$$y_1(t) = 0, \; y_2(t) = y_2(0)\, e^{-t}, \; y_3(t) = -y_2(0)\, e^{-t}, \; y_4(t) = -\frac{3}{\varepsilon} y_2(0)\, e^{-t},$$

and $y_4(0) = -3y_2(0)/\varepsilon$ can be large for small $\varepsilon > 0$.
In contrast, the squared weighted semi-norm of this solution satisfies $\|y(t)\|_E^2 = 2(y_2(0))^2 e^{-2t}$ for $t \geq 0$.

Outline

Lemma (Achleitner, Arnold, M., Nigsch 2023)

Let $J = -J^* \in \mathcal{B}(\mathcal{H})$, $0 \le R = R^* \in \mathcal{B}(\mathcal{H})$. *Consider conditions:*

1. *There exists* $m \in \mathbb{N}_0 \cup \{\infty\}$ *s.t.* $\overline{\mathrm{Span}}(\bigcup_{j=0}^m \mathrm{Im}(J^j \sqrt{R})) = \mathcal{H}$.
2. *There exists* $m \in \mathbb{N}_0 \cup \{\infty\}$ *s.t.* $\bigcap_{j=0}^m \ker(\sqrt{R}(J^*)^j) = \{0\}$.
3. *There exists* $m \in \mathbb{N}_0 \cup \{\infty\}$ *such that* $\forall x \in \mathcal{H} \setminus \{0\}$ $\exists j = j(x) \le m$ *if* m *is finite or* $j = j(x) \in \mathbb{N}$ *if* $m = \infty$ *s. t.* $\langle J^j R (J^*)^j x, x \rangle > 0$.
4. *If a closed subspace* V *of* $\ker R$ *is invariant under* J *then* $V = \{0\}$.
5. *No eigenvector of* J *lies in the kernel of* R.

Then the implications 1. $\Leftrightarrow$ *2.* $\Leftrightarrow$ *3.* $\Leftrightarrow$ *4.* $\Rightarrow$ *5. hold.*
The implication 5. $\Rightarrow$ *4. holds if and only if* $\dim \ker R < \infty$.

▷ F. Achleitner, A. Arnold, V. M., E. Nigsch. Hypocoercivity in Hilbert spaces. http://arxiv.org/2307.08280

Definition (Achleitner, Arnold, M. 2023)

Let $\mathcal{H}$ be a separable Hilbert space. For dissipative operators $A = J - R \in \mathcal{B}(\mathcal{H})$, the hypocoercivity index (HC-index) m_{HC} of A is defined as the smallest integer $m \in \mathbb{N}_0$ (if it exists) such that

$$\sum_{j=0}^{m} (A^*)^j R A^j \geq \kappa I$$

for some $\kappa > 0$.

Actually for bounded operators the HC-index can only be finite.

▷ F. Achleitner, A. Arnold, and V. M. Hypocoercivity in algebraically constrained partial differential equations with application to Oseen equations. In revision. http://arxiv.org/2212.06631

Lemma

Let $J = -J^* \in \mathcal{B}(\mathcal{H})$, $R = R^* \in \mathcal{B}(\mathcal{H})$ *with equal domains and* $\dim\ker R < \infty$. *Then there exists a direct sum decomposition* $\mathcal{H} = \bigoplus_{i=1}^{s} \mathcal{H}_i$ *with* $s \geq 2$ *and each* $\mathcal{H}_i$ *a (possibly zero) subspace of* $\mathcal{H}$ *such that:*

- $\dim \mathcal{H}_i < \infty$ *for* $i = 2, \ldots, s$;
- *with respect to this decomposition,* J, R *are operator matrices of the same form as in the finite dimensional case.*

If a nontrivial last block in the staircase form exists, then the system is not hypocoercive. But in general the number of blocks cannot be used to determine m_{HC}.

Consider the semigroup $P(t) := e^{-At}$ $(t \geq 0)$ and its operator norm $\|P(t)\| := \sup\{\|P(t)x\| : \|x\| = 1\}$.

Theorem (Achleitner, Arnold, M., Nigsch 2023)

Let the operator $A = J - R \in \mathcal{B}(\mathcal{H})$ be dissipative.

a) *If A has index m_{HC}, then there exist $c_1 \geq c \geq c_2 > 0$, $C_1, C_2 \geq 0$ and $t_0 > 0$ such that with $a := 2m_{HC} + 1$ we have*

$$1 - c_1 t^a - C_1 t^{a+1} \leq \|P(t)\| \leq 1 - c_2 t^a + C_2 t^{a+1} \quad \text{for } t \in [0, t_0]$$

and

$$\|P(t)\| = 1 - ct^a + o(t^a) \qquad \text{for } t \to 0.$$

b) *Conversely, if $\|P(t)\|$ satisfies the bounds with some $c > 0$ and $a > 0$, then a is an odd integer and A has index $m_{HC} = \frac{a-1}{2}$.*

▷ F. Achleitner, A. Arnold, V. M., E. Nigsch. Hypocoercivity in Hilbert spaces. http://arxiv.org/2307.08280

Comments

- If $\|P(t)\|$ is sufficiently often differentiable, then we get the same result as in finite dimensional case.
- Open: What are conditions on A so that $\|P(t)\|$ is sufficiently often differentiable (analytic).
- Extension to operator DAEs under investigation.

Outline

Isotropic Oseen on 2D torus

Incompressible Oseen equation with *isotropic* viscosity on 2D torus $\mathbb{T}^2 := (0, 2\pi)^2$,

$$\begin{aligned} u_t &= -(b \cdot \nabla)u - \nabla p + \nu \Delta u\,, t \geq 0\,, \quad \text{on } \mathbb{T}^2\,, \\ 0 &= -\mathrm{div} u\,, t \geq 0\,, \end{aligned}$$

velocity $u(x, t)$, pressure $p(x, t)$ in $x \in \mathbb{T}^2$ and $t \geq 0$.
Assume periodic boundary conditions in both u and p.
Here $\nu > 0$ is the viscosity and $b \in \mathbb{R}^2$ constant drift velocity.
For any initial condition

$$u(0) \in H_{per}(\mathrm{div}0, \mathbb{T}^2) := \{u \in (L^2(\mathbb{T}^2))^2 \,|\, \mathrm{div} u = 0\},$$

the equation has a unique smooth solution for $t > 0$.

Analysis by hypocoercivity

Consider Fourier expansion with

$$u(x,t) = \sum_{k\in\mathbb{Z}^2} \phi_k(t) e^{ik\cdot x}\,, \qquad p(x,t) = \sum_{k\in\mathbb{Z}^2} p_k(t) e^{ik\cdot x}\,.$$

Fourier coefficients $\phi_k(t) \in \mathbb{C}^2$, $p_k(t) \in \mathbb{C}$, $k \in \mathbb{Z}^2$, satisfy decoupled DAEs

$$\begin{cases} \frac{d}{dt}\phi_k &= -ikp_k - i(b\cdot k)\phi_k - \nu|k|^2\phi_k\,, \quad t > 0\,, \\ 0 &= -ik\cdot\phi_k\,. \end{cases}$$

For $k \neq 0$, DAEs exhibit non-trivial dynamics with HC-index 0. The solution $(u(\cdot,t), p(\cdot,t))$ of the Oseen equation converges, as $t \to \infty$, to the constant (in x and t) equilibrium $(\phi_0, p_0) \in \mathbb{R}^3$, for $p_0 = 0$ with the exponential decay rate $\mu = \min_{k\neq 0}(\nu|k|^2) = \nu$.

Anisotropic Oseen equations on 2D torus.

$$\begin{aligned} u_t &= -(b(x)\cdot\nabla)u - \nabla p + \nu\partial_{x_2}^2 u\,,\, t > 0,\ \text{on } \mathbb{T}^2, \\ 0 &= -\mathrm{div} u\,,\, t \geq 0\,, \end{aligned}$$

subject to periodic boundary conditions.
Drift velocity vector $b(x) \in \mathbb{R}^2$ may depend on $x \in \mathbb{T}^2$, and diffusion happens only in x_2.

Consider Hilbert space $\widetilde{\mathcal{H}} := \{u \in \mathcal{H} \mid \int_{\mathbb{T}^2} u dx = 0\}$, endowed with the L^2-inner product.

Proposition

Let $b \in \mathbb{R}^2$ be constant with $b_1 \neq 0$. Then, the operator $b \cdot \nabla - \nu\partial_{x_2}^2$ is neither coercive nor hypocoercive in $\widetilde{\mathcal{H}}$.

Take $b_1 = \sin(x_2)$ and perform a modal decomposition for $k \in \mathbb{Z}^2$. Due to the non-constant coefficient $b_1(x_2)$, these modes decouple now only w.r.t. k_1, but not w.r.t. k_2 and for each mode $k \in \mathbb{Z}^2$ we have the DAE system

$$\begin{cases} \frac{d}{dt}\phi_k &= \frac{k_1}{2}(\phi_{k+e_2} - \phi_{k-e_2}) - ikp_k - \nu k_2^2 \phi_k \ , \quad t > 0 \ , \\ 0 &= -ik \cdot \phi_k \ , \end{cases}$$

Proposition

Let $b_1 = \sin(x_2)$. Then for all $k_1 \in \mathbb{Z} \setminus \{0\}$, the modal dynamics is hypocoercive in $\ell^2(\mathbb{Z};\mathbb{C})$ with HC-index $m_{HC} = 1$.

Outline

Conclusions

- Stability analysis via hypocoercivity, hypocontractivity.
- No spectral information needed in semi-dissipative case.
- Relation to controllability.
- Initial decay rates via HC-index.
- Staircase forms.
- Extension to DAEs and infinite dimensions.
- Analysis for Oseen on 2D torus.
- Open: Hypocoercivity for nonlinear ODEs/DAEs/PDEs

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