Local-to-Global Principles in Floer Theory

Umut Varolgunes

Stanford University

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Hamiltonian Floer theory

- (M, ω) symplectic manifold. (L, J_L) tautologically unobstructed Lagrangian. Assume that they are closed for now.
- Given non-degenerate S^1/I-dependent Hamiltonian H, we obtain chain complexes over $\Lambda_{\geq 0} = \mathbb{Q}[[T^{\mathbb{R}}]]$:

$$CF(H, \Lambda_{\geq 0}) \ / \ CF(L, J_L, H, \Lambda_{\geq 0}) \tag{1}$$

1. generated by 1-periodic orbits / 1-chords on L
2. differential counts Floer solutions with weights $T^{topE(u)}$, where

$$topE(u) = \int u^*\omega + \int_{out} Hdt - \int_{in} Hdt \geq 0$$

Acceleration data

- **Acceleration data** for compact $K \subset M$ is a family of time dependent (S^1 or I) Hamiltonians H_s, $s \in [1, \infty)$ such that:
  1. $H_s(t, x) < 0$, for every t, s and $x \in K$.
  2. $H_s(t,x) \xrightarrow[s \to +\infty]{} \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K, \end{cases}$ for every t
  3. $H_s(t, x) \geq H_{s'}(t, x)$, whenever $s \geq s'$
  4. For $n \in \mathbb{N}$, the flow of H_n satisfies non-degeneracy
- $\mathcal{C}(H_s) := CF(H_1, \Lambda_{\geq 0}) \to CF^*(H_2, \Lambda_{\geq 0}) \to \dots$
  $\mathcal{C}(L, H_s) := CF(L, H_1, \Lambda_{\geq 0}) \to CF^*(L, H_2, \Lambda_{\geq 0}) \to \dots$
- The maps are given by continuation maps. Monotonicity requirement (3) implies that topological energies are all non-negative. These are "1-ray diagrams" over $\Lambda_{\geq 0}$.

Definition of the invariants

- We use the telescope construction of Abouzaid-Seidel as a convenient model for homotopy colimits of 1-rays.
- Completion functor for modules over $\Lambda_{\geq 0}$:

$$A \mapsto \widehat{A} := \varprojlim_{r \geq 0} A \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{\geq r}$$

- $SC_M(K, H_s) := \widehat{tel}(\mathcal{C}(H_s))$ (degree-wise completion, whatever the grading is)
- $LC_M(K, H_s; L) := \widehat{tel}(\mathcal{C}(L, H_s))$
- Homologies are "independent of choices":

$$SH_M(K) \;/\; LH_M(K; L)$$

- Automatically get restriction maps for $K \subset K'$ with the presheaf property

Remarks

- The main point for well-definedness: any Floer theoretic diagram of chain complexes over $\Lambda_{\geq 0}$ can be "filled" to a homotopy coherent Floer theoretic diagram (only monotone choices are allowed).
- Basically a version of Floer-Hofer's symplectic cohomology (the original one)
- Similar invariants by Seidel ("Speculations on pair-of-pants decompositions"), Groman, McLean, Venkatesh (maybe Floer?)
- With M. Abouzaid - Y. Groman: working on extending definition to unobstructed Lagrangians (and their bounding cochains). Significantly harder.

Dependence on K

- $SH_M(\varnothing) = LH_M(\varnothing; L) = 0$ (good exercise)
- $SH_M(M) = H(M, \Lambda_{\geq 0}) \otimes_{\Lambda_{\geq 0}} \Lambda_{>0}$
- $LH_M(M; L) = HF(L, \Lambda_{\geq 0}) \otimes_{\Lambda_{\geq 0}} \Lambda_{>0}$
- If $K \times S^1 \subset M \times T^*S^1$ is displaceable from itself by a Hamiltonian diffeomorphism, then $SH_M(K)$ is torsion.
- If L is displaceable from K by a Hamiltonian diffeomorphism, then $LH_M(K; L)$ is torsion.
- Invariance under symplectomorphisms (given by relabeling choices)

Descent

- Let $F(K)$ denote $SC_M(K, H_s)$ or $LC_M(K, H_s; L)$.
- Floer theory naturally constructs maps

$$F(K_1 \cup K_2) \to cone(F(K_1) \oplus F(K_2) \to F(K_1 \cap K_2)) \qquad (2)$$

We say K_1 and K_2 satisfies descent if this map is a quasi-isomorphism (definition independent of choices).

Theorem (V.)

If K_1 and K_2 admit barriers (a property independent of L) then, K_1 and K_2 satisfy descent.

- For $n > 2$ sets: similar definition for satisfying descent, but theorem requires each pair of finite iterated unions and intersections of K_i's to admit barriers

Involutive systems

- If $\pi : M \to \mathbb{R}^N$ is a smooth involutive map (not surjective), and $C_1, \ldots C_k$ are compact, then $\pi^{-1}(C_1), \ldots, \pi^{-1}(C_k)$ admit barriers. Consequently $F(\pi^{-1}(\cdot))$ is an ∞-sheaf with values in $N_{dg}Ch_{\Lambda_{\geq 0}}$.
- Special cases: Lagrangian fibrations with singularities, compact domains with disjoint boundary (e.g. pair-of-pants decompositions of complex varieties belong here)
- Assume that we are in a situation where for every $C \subset \mathbb{R}^N$, $F(\pi^{-1}(C))$ is non-negatively graded, then

$$C \mapsto H^0(F(\pi^{-1}(C)))$$

is a sheaf (in G-topology of compact sets).

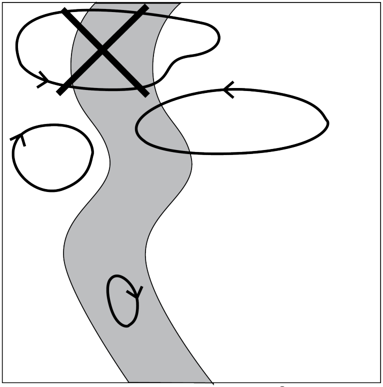
Descent (pf. sketch)

Focus on closed string for now.

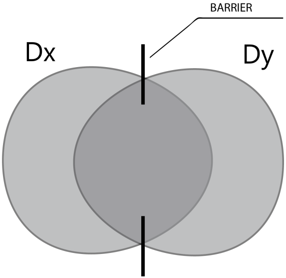
- Let f and g be two non-degenerate Hamiltonians $M \times S^1 \to \mathbb{R}$.
- $U = \{f < g\} \subset M \times S^1$ and $V = \{f > g\} \subset M \times S^1$.
- Assume that $\bar{U}$ and $\bar{V}$ are disjoint. Then, $max(f, g)$ and $min(f, g)$ are smooth functions.
- Assume that no one-periodic orbit of X_f, X_g, $X_{min(f,g)}$ or $X_{max(f,g)}$ has a graph that intersects both U and V. Then,

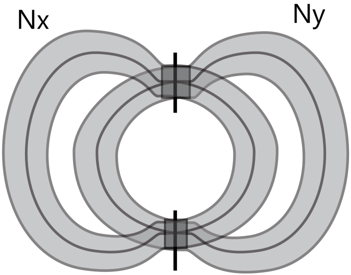
  $$CF(min(f, g)) \to cone(CF(f) \oplus CF(g) \to CF(max(f, g)),$$

  is a quasi-isomorphism
- Use constant solutions for proof



There is an entirely analogous lemma for open string version.





The barrier is a one parameter family of rank 2 coisotropics (coisotropics are unrealistically points in the picture). Main point is that if a coisotropic of any rank belongs to a level set of a Hamiltonian, then it is invariant under its flow. The barrier seperates the flows, hence seperates orbits or chords just the same.

Killing the torsion part (a semi-quantitative version)

- Unless we are interested in questions of quantitative symplectic geometry (displacement energy, capacities etc.), the torsion part is too bulky to carry around.
- Hence, we kill it: $SH_M(K, \Lambda) = SH_M(K) \otimes \Lambda$ (and open string analogue), where $\Lambda = \mathbb{Q}((T^{\mathbb{R}}))$, is the quotient field
- Alternatively, could define Floer groups as vector spaces over Λ from the beginning (just a simple base change), then use completion as normed vector spaces. The norm here is the most stupid one, where each generator has norm 1 (valuation 0). Note continuous=bounded.
- The simplicity of the norm is deceiving. If we assigned arbitrary values as norms of generators, we would obtain an isomorphic Λ-Banach space, as this simply corresponds to a diagonal base change. The real quantitative information is contained in the restriction maps.

Domains with stable boundary: outside generators, you are not wanted

- Here stable is in the sense of a stable hypersurface
- Examples include domains with convex (or concave) boundary
- Stable hypersurfaces admit stable tubular neighborhoods $\Sigma \times (-\epsilon, \epsilon)_r$ such that the Hamiltonian flow of the function r induces the same vector field on each $\Sigma \times \{r\}$.
- For closed string, or $L = \mathcal{L} \times (-\epsilon, \epsilon)$, using an acceleration data that has the form $c \cdot r$, $c \to \infty$ irrational, in portions of the stable neighborhood, we can make sense of "inside" and "outside" generators
- Question 1: can one construct in a natural way a chain complex generated by the inner generators whose homology is $SH_M(K)$? Is it the case that the differential of this complex only sees Floer solutions between inner generators?

Liouville domains: the action rescaling

- Let $W \subset M$ be an embedded Liouville domain with primitive λ
- We assume that Question 1 has an affirmative answer, and pretend that we can simply discard the outside generators
- Do a diagonal change of basis, for which the valuations of the generators become the actions $\int \lambda + \int H dt$. Think of the underlying Λ-Banach space as a completion of $SC_{Vit}(\hat{W}) \otimes_{\mathbb{Q}} \Lambda$ (and $\int H dt$ as negligible - also my periods are negative)
- In this basis the matrix of the differential of $F(W, \Lambda)$ is of the form $d_{loc} + A$, where d_{loc} is a matrix whose elements are in $\mathbb{Q}$ (corresponds to Floer solutions that stay within W) and each entry of A has at least ϵ valuation, for some $\epsilon > 0$.
- CAUTION: this does not mean that A is a deformation in any useful sense as we have ∞-dim vector spaces. We need control on the operator norm on A.

Liouville domains: the action rescaling II

- If we Liouville expand or contract $W \to W_\tau$, then for $F(W_\tau, \Lambda)$, $d = d_{loc} + A$ has exactly the same entries but the norm with respect to which we complete $SC_{Vit}(\hat{W}) \otimes_{\mathbb{Q}} \Lambda$ changes. This relies on "contact Fukaya trick". Bigger size means smaller completion.
- We can think of this as a $(0, S)_\tau$-family ($S > 1$) of Λ-Banach complexes ($\tau = 1$ corresponding to original W).
- One can always artificially extend the family to $[0, S)$, and A is an honest perturbation for $\tau = 0$. We might not be able go further than S as d might stop being a continuous map.
- Question 2: How does the homology of $(F(W_\tau, \Lambda), d)$ vary with $\tau \in [0, S)$? What exactly causes it to change? Is there any stability near 0?

Examples of different τ dependences

- If grow a disk inside a sphere of area A, when the area of the disk reaches $A/2$ suddenly the invariants jumps from zero to non-zero. More complicated example was worked out by Venkatesh.
- Assume $c_1(M) = 0$. If the Liouville domain is index bounded, there is no τ dependence (e.g. disk in T^2).
  A **Giroux divisor** $D = \bigcup D_i$ is an SC divisor with the property that there exist integers $w_i > 0$ and a real number $c > 0$ such that
  $$\sum w_i[D_i] = c \cdot PD[\omega] \; \in \; H_2(M).$$
  McLean's stable displaceability of symplectic divisors, and the Mayer-Vietoris property (for non-intersecting boundaries), leads to rigidity properties of skeleta of complements of Giroux divisors. This is joint work with D. Tonkonog.

Products and unit (w Tonkonog)

- $SH_M(K,\Lambda)$ is a unital algebra. This algebra structure is canonical, associative and commutative (not written down yet).
- $LH_M(K;L,\Lambda)$ is a unital algebra. This algebra structure is canonical and associative (not written down yet).
- Restriction maps are unital algebra homomorphisms.
- There are closed open maps that are unital.
- Vanishing is equivalent to $1=0$.

Nodal fibrations over $\mathbb{R}^2$ (joint w. Y. Groman)

- Take a finite number of points P and from each take a ray in the direction of an integral vector so that they are pairwise disjoint. Denote this data by N.
- N defines an integral affine structure on $\mathbb{R}^2 - P$. The smooth structure on $\mathbb{R}^2 - P$ can be extended to a smooth structure on the entire $\mathbb{R}^2$.
- There is a symplectic manifold M_N and Lagrangian fibration $\pi : M_N \to \mathbb{R}^2$ such that π induces the same integral affine structure on $\mathbb{R}^2 - P$.
- π admits a Lagrangian section Z (zero section away from the critical values) which is the fixed point set of an anti-symplectic involution.
- Using monotonicity techniques via the integrable structure at infinity one can define $HF_{M_N}(K; Z)$ and $SH_{M_N}(K)$.

Non-archimedean mirror of M_N (in progress)

- Starting from N, Kontsevich-Soibelman construct a Λ-analytic space with a non-archimedean torus fibration over $\mathbb{R}^2_N$.
- We can give a more direct construction using $HF^0_{M_N}(\pi^{-1}(C); Z, \Lambda)$, where C's are convex (makes sense!) compact domains containing at most one singular value (small).
- Locality for certain compact subsets of the base via moving worms (similar to Viterbo restriction maps). We can't do this for K3.
- Hartogs, wall-crossing (analyzed through comparison with Family Floer or...) and sheaf property for analyzing the neighborhood of nodal fiber
- Take a cover by small convex domains, construct the topological space by gluing Berkovich spectra and equip with an atlas. Sheaf property can be used to prove that the result is independent of the cover. Its real use is for proving HMS (local generation implies global generation).

Non-archimedean mirror of M_N (in progress)

- Question 3: When is the mirror Stein? Related question: when do restriction maps have dense image (corresponds to the Runge immersions in the mirror)? Assuming M_N also has a compatible Liouville structure, (possibly equivalently) when is it equal to the analytification of $SH_{Vit}(M_N)$? Only log-Calabi Yau?
- I have been trying to prove the density results that are true in the mirror directly using properties of relative Floer theory. Why does the wall crossings and monodromy "cancel each other" for the neighborhood of the nodal fiber? Is it really just because of wall-crossing formula?
- Interesting example $\mathbb{C}P^2-$elliptic curve. Has a compatible Liouville structure. Boundary is a straight line. Mirror definitely not equal to analytification! How does it compare with the AKO mirror, elliptic surface....?