Koç seminar 9/1-2024

Georgios Dimitroglou Rizell, Uppsala University

Floer homology and potentials for Lagrangians with conical singularities

- Previous work w. Ekholm-Tonkonog (dim $2n=4$): the refined potential [arXiv: 1806.03722]
- Current work w. Ghiggini (dim $2n>4$)

Goal: Define and compute Floer homology (an invariant of symplectic topology) for singular Lagrangian subspaces of symplectic manifolds.

- The skeleton Skel $\subseteq (X,\omega)$ of a symplectic (Wein) Stein mfd. is a singular Lagrangian, and should know everything about Floer homologies in X (The skeleton determines X)

- In particular, generate monotone Fukaya categories by the skeleton of divisor complements.

- Homological mirror symmetry: complex variety

  $\underbrace{\text{SYZ-fibration}}_{\text{singular Lagr. torus fibration}}$ on (X,ω) symplectic $\rightsquigarrow$ $\underbrace{(X^\vee, J)}_{\text{space of fibres with loc. sys.}}$

Plan

I. Geometric setting

II. Symplectic invariants: exact case X (Wein)Stein — open symplectic

III. Symplectic invariants: monotone case $\overline{X} = X \cup$ divisor (anti-can) — closed Fano

I. Geometric Setting

The following example will be our main focus today

sympl. 2-form: $d\omega_{FS}=0$, ω_{FS} non-deg 2-form

integrable complex structure

$\omega_{FS}(-,J-)$ is the Fubini-Study metric (Hermitian)

$$(\tilde{X},\omega)=\boxed{(\mathbb{CP}^n,\omega_{FS},J)}$$ is a Kähler manifold.

We're doing symplectic topology, so J is just an auxiliary choice here

"divisor at ∞"

$$\Big(\underbrace{\mathbb{CP}^n\setminus\mathbb{CP}^{n-1}_\infty}_{\mathbb{C}^n},\ \omega_{FS}\Big)\cong\Big(B^{2n},\ \sum dx_i\wedge dy_i\Big)$$

(not as cplx varieties)

linear sympl. 2-form

$L^n\subset(X^{2n},\omega)$ Lagrangian if $\boxed{\omega|_{TL}\equiv 0}$

($\Rightarrow \dim L\le n$

in general: isotropic submfd)

In Kähler mfds:

L Lag $\Leftrightarrow$ $TL\perp JTL$ & $\dim L=n$

$\Rightarrow TL \pitchfork JTL$ (tot. real)

Ex $(\mathbb{CP}^1,\omega_{FS})\cong(S^2,\sigma_{area})$

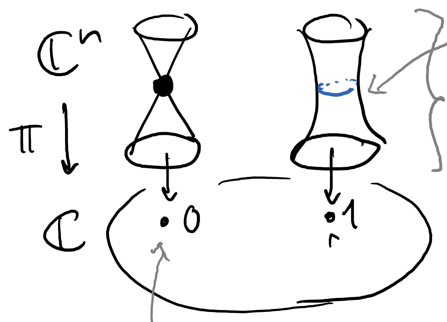
area form

any curve is automatically Lagrangian!

Removing an anti-canonical divisor

Since $(B^{2n}, dx_i\wedge dy_i)$ has no non-trivial Floer homology (it is a Stein domain whose skeleton = a point), we want to cut out more from $\mathbb{CP}^n$: the anti-canonical divisor.

$\pi: z_1\cdots z_n : \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}_\infty \longrightarrow \mathbb{C}$ holomorphic (cplx. polynomial)



Lagrangian $\mathbb{T}^{n-1} = (S^1)^{n-1} \subseteq (\mathbb{C}^*)^{n-1}$

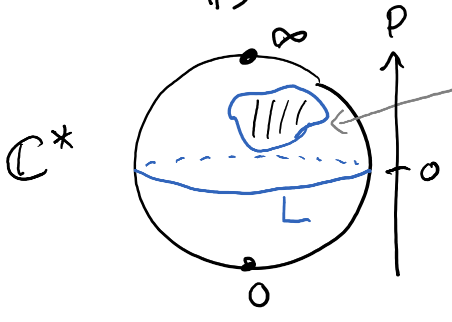
generic fibre is bihol. to $(\mathbb{C}^*)^{n-1}$
(all $z_i \neq 0$)

symplectomorphic to $\left(D^*\mathbb{T}^{n-1}_\theta, \sum d(p_i d\theta_i)\right)$
$= \left(D^*S^1_\theta, d(p\,d\theta)\right)^{n-1}$

unique singular value $= 0$
preimage = union of n hyperplanes $\{z_i = 0\}$
(non-singular if $n=1$, isolated sing iff $n=2$)

<u>Ex</u> $\pi^{-1}(1) \overset{n=2}{=} \left(D^*S^1, d(p\,d\theta)\right) = \left(S^1_\theta \times [-\tfrac{1}{2}, \tfrac{1}{2}]_p, dp\wedge d\theta\right)$

$\cong$



non-exact (bounds a hol. disc)

$L = \{p = 0\} \subseteq D^*S^1$ is an exact Lag
i.e. $p\,d\theta|_{TL}$ is an exact one-form

We write $\boxed{X^n = \mathbb{CP}^n \setminus \underbrace{(\mathbb{CP}^{n-1}_\infty \cup \pi^{-1}(1))}_{\text{divisor of degree } n+1 \Rightarrow \text{anti-canonical divisor}}}$

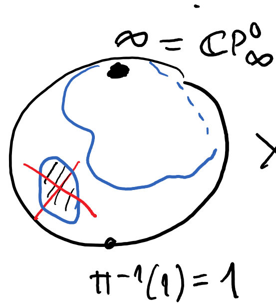
This is a (Wein) Stein manifold with more interesting Lagrangians (and skeleton) than $B^{2n} \supseteq X^n$

Smooth Lagrangian tori in X

The weakly exact Lagrangians (bound no holomorphic discs) have non-trivial Floer-theoretic invariants

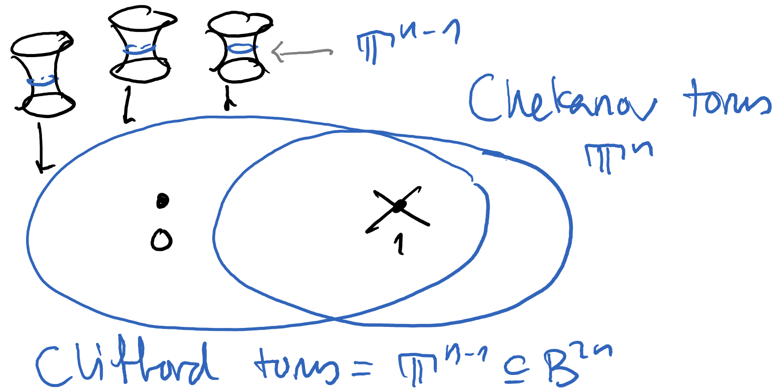
When $n=1$: $\{$weakly exact Lag's$\}$ =

$\{$non-contractible closed curve $\subseteq \mathbb{C}^*\}$



$X = \mathbb{C}^*$

$n \geq 2$: All known weakly exact Lag's are of the form



Lagrangian torus $\mathbb{T}^n \subseteq X$ is given by parallel-transp. of the Lagr. torus $\mathbb{T}^{n-1} \subseteq \pi^{-1}(pt)$

$n=2$: [DR] classified all (weakly) exact Lagrangian tori: they are of the above form [arXiv: 1712.01182]

Singular Lagrangians

Lagrangians w. "arboreal singularities" (due to [Nadler])

$\cap\!|$

Skeleta of Weinstein mfd^s

$\|$

Lagrangians w. conical singularities

link of singularity

$$\left((-\infty,0]_z \times Y^{2n-1}, d(e^z\alpha_Y)\right) \supseteq \underbrace{(-\infty,0] \times \Lambda^{n-1}}$$

symplectic (non-deg) $\Leftrightarrow$ $\alpha_Y \in \Omega^1(Y)$ contact form

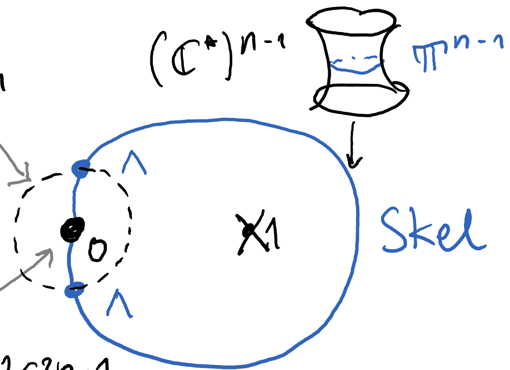
Lagr. iff $\Lambda^{n-1} \subseteq (Y^{2n-1}, \alpha_Y)$ is Legendrian, i.e. $\alpha_Y|_{T\Lambda} \equiv 0$

Obs Today the ambient sympl. mfd is smooth, namely there $\exists$ a smooth compactification $(-\infty,0] \times Y$.

The skeleton of $X^n = \mathbb{C}P^n \setminus (\mathbb{C}P^{n-1}_\infty \cup \pi^{-1}(1))$

$Y = \partial D^{2n} = S^{2n-1}$

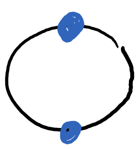
$$\left((-\infty,0]_z \times \overset{\overset{Y}{\|}}{S^{2n-1}}, d(e^z\alpha)\right) \subseteq (D^{2n}, dx\wedge dy)$$

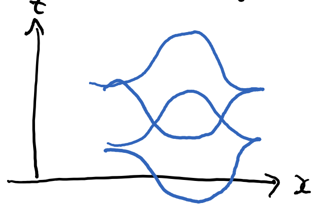


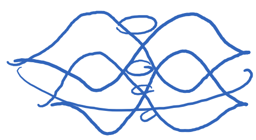
$$\alpha = \frac{1}{2}\sum (x_i dy_i - y_i dx_i)\Big|_{TS^{n-1}} \in \Omega^1 S^{2n-1}$$

- Skel has an isolated singularity at 0

- The Legendrian link of the singularity:

$n=1$  $Y = S^1$

$n=2$ 

$n=3$ 

std. Legendrian Hopf-link.

- Skel $\setminus 0 = I \times \mathbb{T}^{n-1}$

II. Symplectic invariants in X (exact case)

The Chekanov-Eliashberg DGA of the Legendrian Λ.

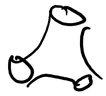
$$\mathcal{A}_\Lambda := C_*(\Omega(\mathrm{Skel}\setminus 0);\mathbb{C})\,\langle \text{Reeb chords on } \Lambda\rangle$$

sub-DGA (non-central coeff.) free generators

This is free non-comm. DGA whose quasi-isomorphism class is invariant under Legendrian isotopy.

(as DGAs)

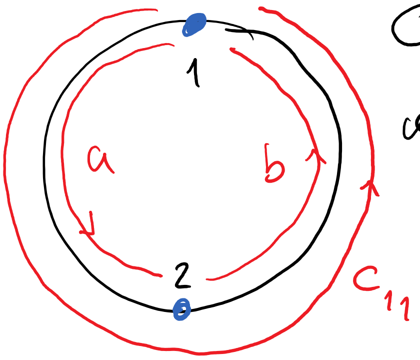
$$C_*(\Omega(\mathrm{Skel}\setminus 0);\mathbb{C}) \simeq C_*(\Omega T^{n-1};\mathbb{C}) \simeq \mathbb{C}[\mu_1^{\pm1},\dots,\mu_{n-1}^{\pm1}]$$

regular functions on a conic bundle that degenerates over 

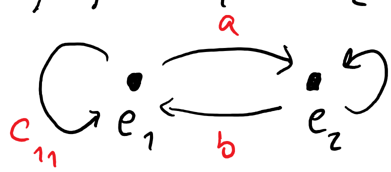
Thm $\mathcal{A}_\Lambda \simeq \mathbb{C}[\mu_1^{\pm1},\dots,\mu_{n-1}^{\pm1},u,v]/uv=\mu_1+\dots+\mu_{n-1}+1$

$\sum \mu_i = -1$ pair of pants $\subseteq (\mathbb{C}^*)^{n-1}$

Ex $\boxed{n=1}$ gives $\mathcal{A}_\Lambda \simeq \mathbb{C}[u^{\pm1}] \simeq C_*\Omega(S^1)$



$C_*(\Omega\Lambda;\mathbb{C}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ semi-simple alg

$\mathcal{A}_\Lambda =$ 

infinitely gen. free DGA

$\partial c_{11} = ab - 1$

$\vdots$

See [Etgü-Lekili] or [Bäcke] for computation

- The surgery formula by [Bourgeois-Ekholm-Eliashberg]

  The wrapped Fukaya category $\mathrm{WFuk}(X) \simeq \mathrm{Perf}\,\mathcal{A}_\Lambda \subseteq \mathcal{A}_\Lambda\text{-mod}$

  (Previously computed by [Abouzaid-Sylvan] via different methods)

- The sub category $\mathrm{Fuk}(X) \subseteq \mathrm{WFuk}(X)$ generated by compact Lagrangians embeds into $\mathrm{Prop}\,\mathcal{A}_\Lambda \subseteq \mathrm{Perf}\,\mathcal{A}_\Lambda$

  finite-dimensional $\mathcal{A}_\Lambda$-modules

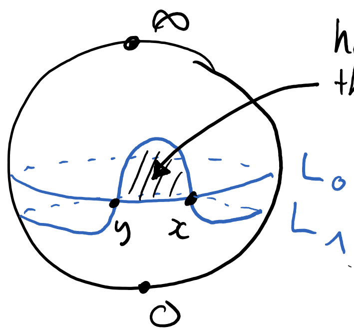
Floer homology in X

$L_0, L_1 \subseteq X$ smooth Lagrangians equipped with local systems $\mathcal{L}_i$

$CF_\bullet(L_0, L_1) = \bigoplus_{x \in L_0 \pitchfork L_1} \mathbb{C} \cdot x$ $\quad$ $\mathbb{C}$-vector space spanned by $L_0 \pitchfork L_1$

Ex $CF_\bullet(\text{Clifford}, \text{Clifford})$

$\simeq H_\bullet(S^1)$ (as DGA's)



holomorphic strip that contributes to $\langle \partial x, y \rangle$

We now define the Floer homology of the Skeleton with coefficients in $\boxed{\mathcal{A}^e_\Lambda := \mathcal{A}_\Lambda \otimes_{\mathbb{C}} \mathcal{A}^{op}_\Lambda}$

i.e the free $\mathcal{A}_\Lambda$- bimodule of rank 1

also called the universal 2-sided local system on Skel

$$CF_\bullet(\text{Skel}_0, \text{Skel}_1; \mathcal{A}^e_\Lambda) = \bigoplus_{\substack{x \in \text{Skel}_0 \pitchfork \text{Skel}_1 \\ \text{or Reeb ch. from } \Lambda_0 \to \Lambda_1}} \mathcal{A}^e_\Lambda \cdot x$$

Thm [Chantraine - DR - Ghiggini]

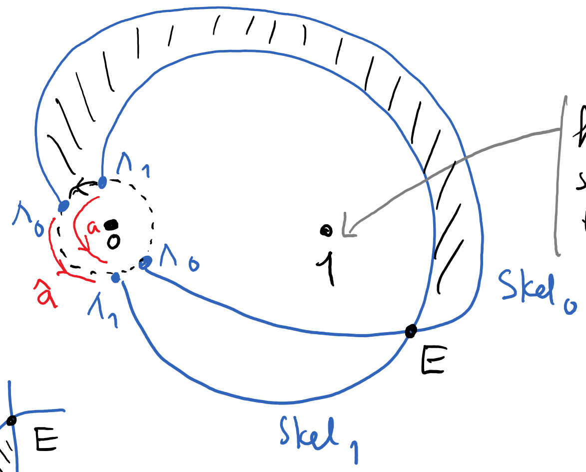
$$\underbrace{CF_\bullet(\text{Skel}, \text{Skel}; \mathcal{A}^e_\Lambda)}_{\substack{\text{a semi-free } \mathcal{A}_\Lambda\text{-bimodule} \\ (\Leftrightarrow \mathcal{A}^e_\Lambda\text{-module})}} \simeq \mathcal{A}_\Lambda$$

the diagonal $\mathcal{A}_\Lambda$- bimodule (not a free $\mathcal{A}^e_\Lambda$-module)

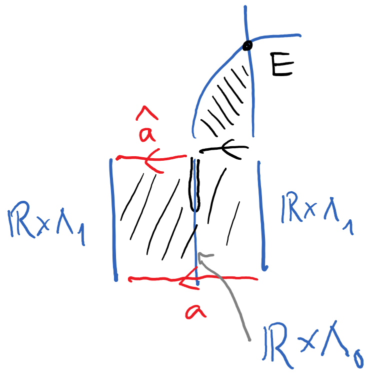
Also c.f. [Legout, arXiv: 2304.03014]

Floer homology of the skeleton

Ex



fibre over 1 is removed so no strips can pass through here



this contributes to $\partial\hat{a} = \boxed{(a\otimes 1)\cdot E} - (1\otimes a)\cdot E$

(similarly $\partial\hat{b} = (b\otimes 1)\cdot E - (1\otimes b)\cdot E$)

- $(L_i, \mathcal{L}_i) \longmapsto \mathcal{M}_i$ finite-dimensional $\mathcal{A}_\Lambda$-module

$$\boxed{CF_\cdot(L_0, L_1) \simeq CF_\cdot(\mathrm{Skel}, \mathrm{skel}; \mathcal{M}_1 \otimes_{\mathbb{C}} \mathcal{M}_0^\vee)}$$

$$= CF_\cdot(\mathrm{Skel}, \mathrm{Skel}; \mathcal{A}_\Lambda^e) \otimes_{\mathcal{A}_\Lambda^e} \underbrace{(\mathcal{M}_1 \otimes_{\mathbb{C}} \mathcal{M}_0^\vee)}_{\substack{\mathcal{A}_\Lambda\text{-bimodule} \\ \text{i.e. } \mathcal{A}_\Lambda^e\text{-module}}}$$

i.e. Fuk(X) embeds into the category of "finite-dim^l local systems on Skel".

- BEE's surgery formula provides: the diagonal bimodule

$$\mathcal{S}: \underset{\substack{\text{symplectic homology} \\ \text{(closed orbit Floer thy.)}}}{SC_\cdot(X)} \xrightarrow{\cong} CF_\cdot(\mathrm{Skel}, \mathrm{Skel}; \mathcal{A}_\Lambda) = CF_\cdot(\mathrm{Skel}, \mathrm{Skel}; \mathcal{A}_\Lambda^e) \otimes_{\mathcal{A}_\Lambda^e} \mathcal{A}_\Lambda$$

$$= CC_\cdot(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda) \text{ Hochschild complex}$$

Expectation: the following diagram commutes / homotopy

$$\begin{tikzcd}
\oint : SC_\bullet(X) \arrow[r, "\simeq"] \arrow[dr, "\mathcal{CO}"'] & CC_\bullet(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda) = CF_\bullet(\mathrm{Skel}, \mathrm{Skel}; \mathcal{A}_\Lambda) \arrow[d, "\simeq"', "\text{ey}"] \\
 & CC^{n-\bullet}(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda) \overset{\Phi}{\simeq} CF^{n-\bullet}(\mathrm{Skel}, \mathrm{skel}; \mathcal{A}_\Lambda)
\end{tikzcd}$$

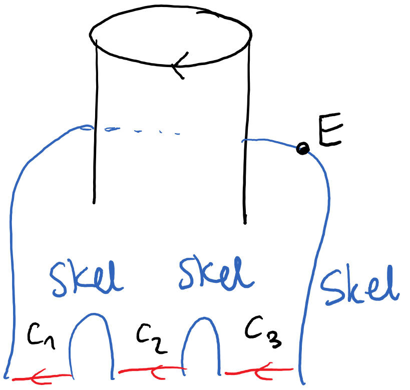
?

Calabi-Yau q.i [Legout]

Hochschild co-complex
(constructed using bar resolution)

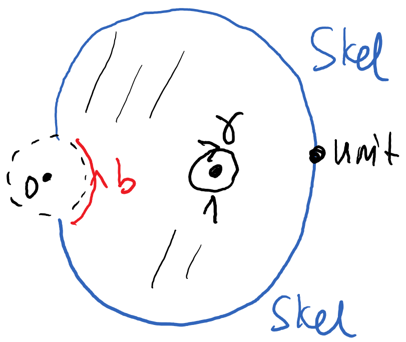
closed open map, this is a quasi-isomorphism by [Ganatra]

$\gamma \in SC_\bullet(X)$



$\langle \oint(\gamma), E \rangle = c_1 c_2 c_3$

$\phi \circ \mathcal{CO} =: \widetilde{\mathcal{CO}}$



$\widetilde{\mathcal{CO}}(\gamma) = b \cdot \mathrm{unit} \in CF^\bullet(\mathrm{Skel}, \mathrm{Skel}, \mathcal{A}_\Lambda)$

Also, c.f. recent work by [J. Smith], whose work is related to the case when $\mathrm{Skel} \subseteq X$ is smooth, e.g:

$$X = \mathbb{CP}^n \setminus \left(\mathbb{CP}^{n-1}_\infty \cup \pi^{-1}(0)\right)$$

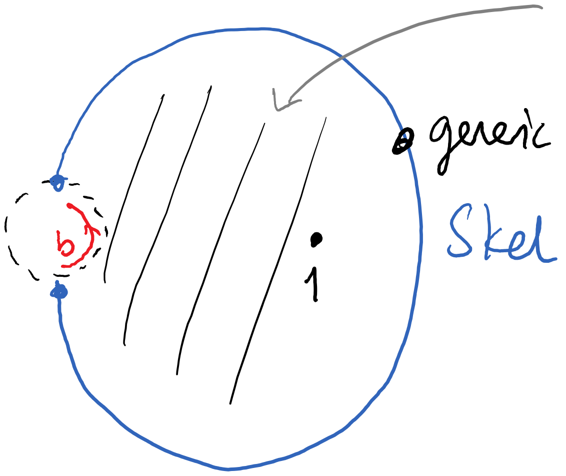
$$\mathrm{Skel}(X) = \text{Clifford } \mathbb{T}^n \quad (\text{smooth})$$

$$\mathcal{A}_\Lambda \simeq \mathbb{C}[\mu_1^{\pm 1}, \dots, \mu_n^{\pm 1}] \simeq C_\bullet \Omega \mathbb{T}^n$$

III. Symplectic invariants: $\overline{X}$ the monotone case

Now we complete X to $\overline{X} = \mathbb{CP}^n$ by adding back the divisor. Now there are discs/Floer passing through the divisor.

We define the <u>refined potential</u> by counting "big" holomorphic discs in $\overline{X}$ with boundary on $\mathrm{Skel} \subseteq X$ (they are allowed to have ends on the singularity, but must pass through the divisor $\overline{X} \setminus X$)



disc contributing to the term

$\underline{n=1}$:

$$\mathcal{P}_{\mathrm{Skel}} = [a + b] \in H_\cdot(\mathcal{A}_\Lambda)$$

$$\| \qquad\qquad \|$$

$$u + u^{-1} \in \mathbb{C}[u^{\pm 1}]$$

Thm $\mathcal{P}_{\mathrm{Skel}} = u + \dfrac{v^n}{\mu_1 \cdots \mu_{n-1}} \in H_\cdot(\mathcal{A}_\Lambda) = \dfrac{\mathbb{C}[u, v, \mu_i^{\pm 1}]}{uv = \mu_1 + \cdots + \mu_{n-1} + 1}$

We recover the classical potentials by <u>localisation</u>

adjoining

$\boxed{v^{-1}}$: $[\mathcal{P}_{\mathrm{Skel}}] = \mathcal{P}_{\mathrm{Clifford}} \in H_\cdot(\mathcal{A}_\Lambda)[v^{-1}] = \mathbb{C}[v^{\pm 1}, \mu_1^{\pm 1}, \ldots, \mu_{n-1}^{\pm 1}] \simeq C_\cdot \Omega T^n$

$\boxed{u^{-1}}$: $[\mathcal{P}_{\mathrm{Skel}}] = \mathcal{P}_{\mathrm{Chekanov}} \in H_\cdot(\mathcal{A}_\Lambda)[u^{-1}] = \mathbb{C}[u^{\pm 1}, \mu_1^{\pm 1}, \ldots, \mu_{n-1}^{\pm 1}] \simeq$

Floer homology in the monotone case

$CF_\bullet^{\overline{X}}(\text{Skel}, \text{Skel}; H_\cdot \mathcal{A}^e_\Lambda)$ in $\overline{X}$ is a deformation of

$CF_\cdot(\text{Skel}, \text{Skel}; H_\cdot \mathcal{A}^e_\Lambda) \simeq H_\cdot \mathcal{A}_\Lambda$ (since $\mathcal{A}_\Lambda$ formal)

Problem: $\partial^2 x = (P_{\text{Skel}} \otimes 1 - 1 \otimes P_{\text{Skel}}) \cdot x$
(it is no longer a complex)

However: $CF_\cdot^{\overline{X}}(\text{Skel}, \text{Skel}; H_\cdot \mathcal{A}_\Lambda) \simeq CF^{n-\cdot}_{\overline{X}}(\text{Skel}, \text{Skel}; H_\cdot \mathcal{A}_\Lambda)$

are complexes. They are deformations of $CC_\cdot(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda)$

& $CC^\cdot(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda)$ by Floer strips that intersect the divisor.

Expectations: (at least when $\mathcal{A}_\Lambda \simeq$ affine algebra)

- $CF^\bullet_{\overline{X}}(\text{Skel}, \text{Skel}; H_\cdot \mathcal{A}_\Lambda)$ computes the Jacobi algebra $\mathcal{A}_\Lambda / \langle \partial_u P_{\text{Skel}}, \partial_v P_{\text{Skel}}, \partial_{M_i} P_{\text{Skel}} \rangle$ (Diff^{l} of $CF^\cdot$ have terms corr to differentiating $\widetilde{\mathcal{O}\mathcal{C}}(BS) = P_{\text{Skel}} \cdot$ unit)
- $\widetilde{\mathcal{C}\mathcal{O}} : QH^\cdot(\overline{X}) \to CF^\bullet_{\overline{X}}(\text{Skel}, \text{Skel}; H_\cdot \mathcal{A}_\Lambda)$ is a q.is.

  (use q.is in the exact case, see section II, and combine with [Borman-Sheridan-Varolgüneş])

- [Cho-Hong-Lau]'s functor $(L, \mathcal{L}) \mapsto CF^\bullet_{\overline{X}}(\text{Skel}, (L, \mathcal{L}))$ $\in MF(\mathcal{A}_\Lambda, P_{\text{Skel}} - \lambda)$
  ($\lambda = \#\, \mu = 2$ discs)

  yields an embedding of the monotone Fukaya category into the category of Matrix factorisations of $(\mathcal{A}_\Lambda, P_{\text{Skel}})$