# A journey into the world of $p$-adic heights

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*based on joint projects with*
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Koç University
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⋊ **Question 2**. How can one compute them numerically?

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- time to time, I'll cheat a bit...

# Overview

1. Preliminaries
   - $p$-adic numbers
   - Elliptic curves and abelian varieties
   - Curves and their Jacobians
2. Classical Heights
   - Néron–Tate height pairing
   - Birch and Swinnerton–Dyer (BSD) conjecture
3. $p$-adic Heights & Our Results
   - Motivation: $p$-adic BSD and quadratic Chabauty
   - Coleman–Gross height pairing
   - Mazur–Tate height pairing
   - Schneider height pairing
4. Future Work
   - A dream: quadratic Chabauty at bad primes
   - $p$-adic BSD
5. Acknowledgements & References

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**Example**. For $n \in \mathbb{Z}$,

$$v_p(p^n) = n, \quad |p^n|_p = p^{-n}.$$

Then $|p^n|_p$ is small when $n$ is large; $p^n \to 0$ ($p$-adically!).

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*There is a general philosophy in Number Theory that "all completions are created equal" and should have the same rights.* - M. Stoll

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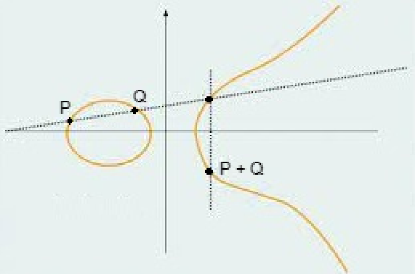

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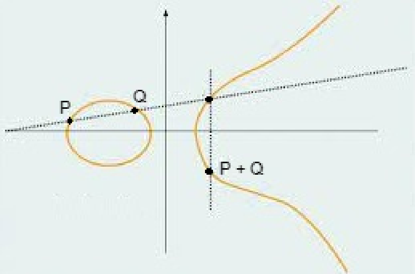


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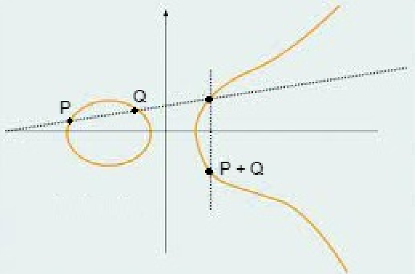


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*It is possible to write endlessly on elliptic curves. (This is not a threat.)* - S. Lang

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$$A(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus A(\mathbb{Q})_{\mathrm{tors}}$$

for some $r \in \mathbb{Z}_{\geq 0}$, called the **algebraic rank** of $A/\mathbb{Q}$.

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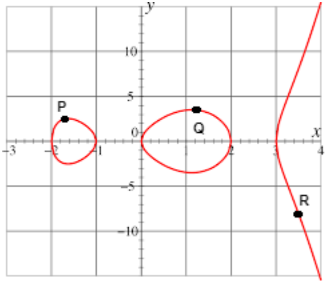


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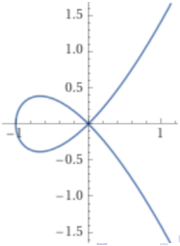

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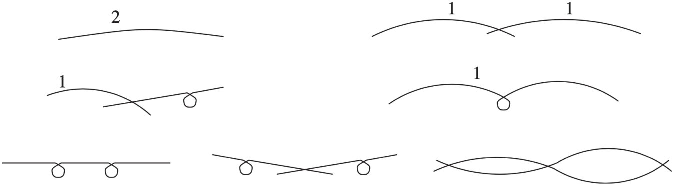


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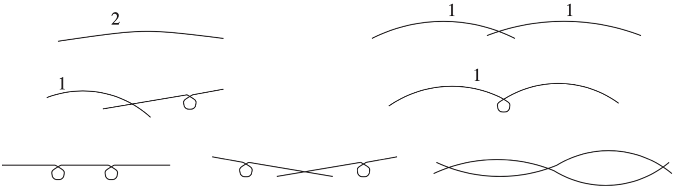


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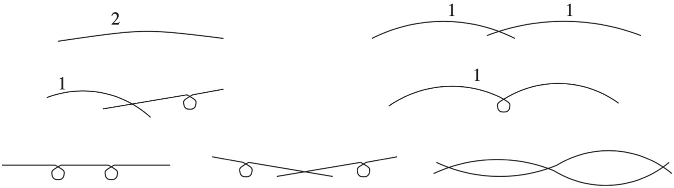


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*Algebraic curves were created by God and algebraic surfaces by the Devil.*
- M. Noether

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$$\text{points in J} \;=\; \text{Div}^0(C)/\sim .$$

The variety $J$ is called the **Jacobian variety** of $C$. For a fixed point $O \in C$, we have an embedding

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# §1.3. Jacobians of curves

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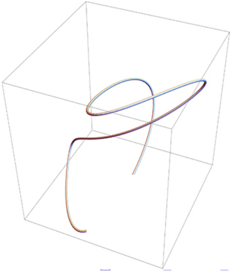

# §2.1. Size of points

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# §2.1. Néron–Tate height pairing

The naive height is

$$h_{\text{naive}} \colon \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}, \quad x \mapsto \log\Big(\max\big\{|x_0|, |x_1|, \ldots, |x_n|\big\}\Big)$$

where $x = (x_0 : x_1 : \cdots : x_n)$, $x_i \in \mathbb{Z}$ and $\gcd(x_0, x_1, \ldots, x_n) = 1$.

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# §2.2. BSD conjecture for abelian varieties

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Here, $\text{Reg}(J/\mathbb{Q})$ is the canonical regulator, i.e.,

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*This remarkable conjecture relates the behavior of a function L, at a point where it is not at present known to be defined, to the order of a group Ш, which is not known to be finite.* - J. Tate

# §3.1. $p$-adic heights and motivation

Fix a prime number $p$. A $p$-adic height pairing is a function

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The $p$-adic height pairing constructed by Schneider is particularly important because

- the corresponding $p$-adic regulator fits into $p$-adic versions of Birch and Swinnerton-Dyer conjecture.

# §3.2. Coleman–Gross height pairing

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$$\langle D_1, D_2 \rangle_p^{\mathrm{CG}} := {}^{\mathrm{Vol}}\!\!\int_{D_2} \omega_{D_1}$$

where

- $\omega_{D_1}$ is a "canonical" differential form attached to $D_1$, and
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**Example**: Consider the hyperelliptic curve $X/\mathbb{Q}_5$ given by

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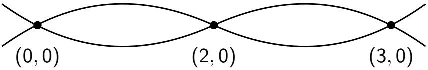

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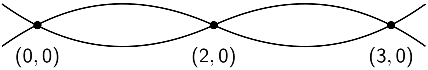


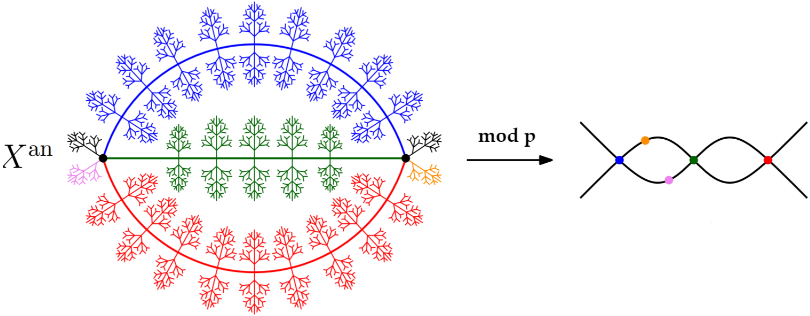

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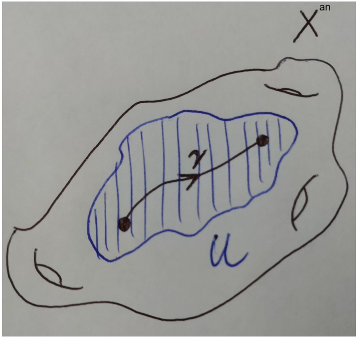

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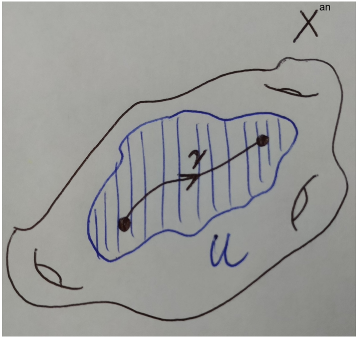


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**Theorem (Katz–K, K.)**

*We have*

$$ {}^{\mathrm{Vol}}\!\int_P^Q \omega = {}^{\mathrm{BC}}\!\int_\gamma \omega - \sum_i \left( c_i \cdot {}^{\mathrm{BC}}\!\int_{\gamma_i} \omega \right) $$

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## Algorithm (Katz–K, K.)

*Compute Vologodsky integrals on hyperelliptic curves using this formula and the fact that the Berkovich–Coleman integral is local.*

# §3.2. Computing Coleman–Gross height pairing

An algorithm to compute the local heights away from $p$ was provided by Müller. **Remark**. A different, but similar, algorithm was developed independently by Holmes.

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Determining $\omega_{D_1}$ from $D_1$ is tricky...

# §3.3. Mazur–Tate height pairing

The pairing

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so one can compute the global Mazur–Tate height pairing.

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The real-valued Néron–Tate height pairing $\langle\cdot,\cdot\rangle^{\mathrm{NT}}$ can be decomposed into a sum of "Néron" functions. Here is a $p$-adic analogue:

# §3.3. Mazur–Tate height pairing

**Theorem (Bianchi–K.–Müller)**

*For each q, there exists a p-adic "Néron function" $\lambda_q$ such that the local pairing $\langle \cdot, \cdot \rangle_q^{\mathrm{MT}}$ can be expressed in terms of $\lambda_q$.*

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**Algorithm (Bianchi–K.–Müller)**

*Compute the Mazur–Tate p-adic height pairing on Jacobians of hyperelliptic curves of genus 2 using the "theta expression" of $\lambda_q$.*

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# §3.4. Mumford curves

**Definition**. $C/\mathbb{Q}_p$ is called a **Mumford curve** if $C$ mod $\mathbb{F}_p$ is a union of curves of genus 0.

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Assume from now on $C/\mathbb{Q}_p$ is a Mumford curve. Then it admits a $p$-adic uniformization: there exists a $p$-adic "Schottky" group $\Gamma$ together with a "$p$-adic analytic" isomorphism

$$C \simeq \Omega/\Gamma$$

where $\Omega = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$, the $p$**-adic upper half plane**.

Fix two parameters $a, b \in \Omega$, and define the **theta function** on $\Omega$:

$$\Theta(a, b; z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)}, \quad z \in \Omega.$$

This is a remarkable "automorphic" form.

# §3.4. Werner's formula for $\langle \cdot, \cdot \rangle_p^{\mathrm{Sch}}$

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$$D = (x) - (y) \quad \text{and} \quad E = (z) - (w)$$

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**Theorem (Werner).** Choose preimages $x', y', z', w'$ in $\Omega$. We then have

$$\langle D, E \rangle_p^{\mathrm{Sch}} = \log_p \left( \frac{\Theta(x', y'; z')}{\Theta(x', y'; w')} \right) - \begin{array}{c}\text{another function}\\ \text{in terms of } \Theta.\end{array}$$

# §3.4. Computing Schneider height pairing

**Algorithm (K.–Masdeu–Müller–van der Put (in progress))**

*Compute the Schneider $p$-adic height pairing on Jacobians of hyperelliptic Mumford curves.*

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**Theorem (K.–Masdeu–Müller–van der Put)**

$$H(z) := \Theta(a, \gamma(a); z) \cdot \prod_{i=0}^{g} \Theta(a_i, b; z) \cdot \Theta(b_i, s_0(b); z), \quad z \in \Omega$$

*is such a function.*

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*There are no bad primes, really.* - N. Dogra

# §4.2. $p$-adic BSD for abelian varieties

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On the other hand, the MTT conjecture in the case of split multiplicative reduction, the "exceptional" case, is of special interest. One might expect that a generalization of this conjecture to higher dimensional abelian varieties in the case of "split purely toric" reduction can be formulated.

# with Katz, Columbus (2019)

# with Masdeu and Müller, Benasque (2022)

# with Bianchi and Müller, Groningen (2023)

# The journey has ended :-) Teşekkürler!

- *Diophantine Geometry: An Introduction* - Hindry–Silverman
- *Fundamentals of Diophantine Geometry* - Lang

- *p-adic heights on curves* - Coleman–Gross
- *Canonical height pairings via biextensions* - Mazur–Tate
- *p-adic height pairings I* - Schneider
- *Local Heights on Mumford Curves* - Werner
- *Computational tools for quadratic Chabauty* - Balakrishnan–Müller

- *Algorithms for Schneider heights on Mumford curves* (in progress) - K.–Masdeu–Müller–van der Put
- *Algorithms for Coleman–Gross Heights on Hyperelliptic Curves* (in preparation) - Bianchi–K.–Müller
- *Coleman–Gross heights and p-adic Néron functions on Jacobians of genus 2 curves* - Bianchi–K.–Müller
- *Explicit Vologodsky integration for hyperelliptic curves* - K.
- *p-adic Integration on bad reduction hyperelliptic curves* - Katz–K.