A Morse-Bott approach to the equivariant homotopy type

joint work with Laurent Côté

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Overview

Floer homology: motivation

Floer homology: an umbrella term for invariants defined using “Morse theory” in “infinite dimensions”.

- Floer defined invariants in symplectic/low dimensional topology
- hugely successful: e.g. solution to Arnold conjecture
- shapes the modern symplectic topology: the central focus
- many applications in dynamics
- gives rise to (half of) a mathematical formulation of mirror symmetry phenomenon in physics
- connects symplectic topology with other fields including algebraic geometry, algebraic topology (string topology), topological data analysis

Floer homology: refinements

Cohen-Jones-Segal (1995): introduce a framework to refine Floer theoretic invariants

$$\text{Space} \longrightarrow \text{Homology groups}$$

$$\text{“Floer homotopy”} \longrightarrow \text{Floer homology}$$

This allows one to define Floer type of invariants for any generalized homology theory (e.g. K-theory).

Equivariant case?

The setup of Floer homotopy theory is not compatible with group actions.

Main goals:

- extend this framework in order to define equivariant Floer homotopy type
- define equivariant and homotopical versions of major invariants in symplectic topology
- an upgrade to well-known Viterbo isomorphism theorem
- applications/further directions:
  - recovering back information about the homology of the manifold from Floer theory
  - Coulomb branches (related to conjectures in physics)
  - ... (more at the end)

Review of Morse theory

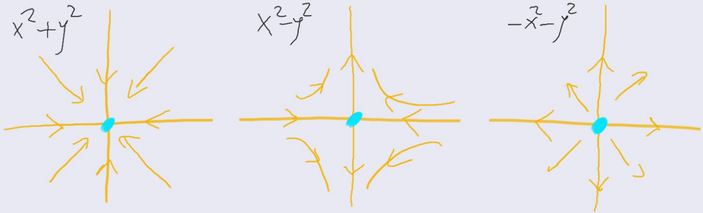
Floer theory: Morse theory in infinite dimensions

Morse theory: study of the manifold topology using functions and their gradient flow

- functions can be complicated, restrict to a special class: Morse functions
- a Morse function f has isolated critical points
- near these, f looks like $f = c \pm x_1^2 \pm x_2^2 \cdots \pm x_n^2 : \mathbb{R}^n \to \mathbb{R}$

Example

The flow of $-grad(f)$ for $f = x^2 + y^2$, $x^2 - y^2$ and $-x^2 - y^2$.



Review of Morse theory (cont'd)

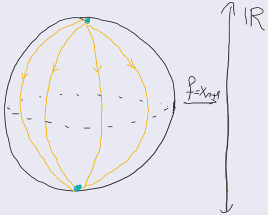
More generally: A *Morse function* $f : N^n \to \mathbb{R}$ is obtained by patching together functions of this form.

Gradient flow: Given metric g, one can define a vector field $grad_g(f)$

- the gradient flow leads to handlebody decompositions (classical)
- gives a way to compute homology (classical)

Example

$N = S^n = \{x_1^2 + \dots x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$, $f = x_{n+1}$.



Longitudes show the flow lines of $-grad_g(f)$.

Given $x, y \in crit(f)$, let $\mathcal{M}_f^\circ(x,y)$ be the moduli of gradient trajectories from x to y (modulo translation).
$\mathcal{M}_f^\circ(x,y)$ is a smooth manifold (for generic g), and admits a compactification $\mathcal{M}_f(x,y)$ into a manifold with corners obtained by adding broken gradient trajectories.

In this picture, $\mathcal{M}_f^\circ(x,y) \cong (-1,1)$, $\mathcal{M}_f(x,y) \cong [-1,1]$

From Morse to Floer

Morse homology: a way to compute homology of N using the data of critical points and $0, 1$ dimensional moduli $\mathcal{M}_f(x, y)$
This data is all we need to define a Floer theory:

Example

"Morse homology" on the free loop space $\mathcal{L}M = Map(S^1, M)$ (M is an exact symplectic manifold) is called *symplectic cohomology*, and denoted by $SH(M)$. It is an example of a Floer homology theory. Its definition requires only the knowledge of critical points of an action functional $\mathcal{A} : \mathcal{L}M \to \mathbb{R}$ and $0, 1$ dimensional moduli of gradient trajectories.

It is a very useful invariant: can detect exotic symplectic manifolds, etc. Note that symplectic cohomology is a generalization of quantum cohomology to non-compact manifolds.

Recovering the homotopy type from Morse theory

Cohen–Jones–Segal: provide a framework to compute (stable) homotopy type using all $\mathcal{M}_f(x, y)$

Example (Large 2021)

Cohen–Jones–Segal's construction on $\mathcal{L}M$ leads to a (stable) homotopy type $SH(M, \mathbb{S})$, whose homology is $SH(M)$. We also refer this as *the symplectic cohomology*. This is also useful (can distinguish symplectic manifolds).

Remark

The $SH(M)$ is highly structured (a framed E_2 algebra) and $SH(M, \mathbb{S})$ is expected to be even more structured (a framed E_2-algebra and also a cyclotomic spectrum). The simplest bit of the expected structure is a circle action.

Equivariant case

Observe: $\mathcal{L}M$ carries an S^1-action (loop rotation). Can one induce an action on $SH(M, \mathbb{S})$?
Unfortunately, Morse theory is not compatible with group actions, even in finite dimensions: assume N carries action of a compact Lie group G. Typically, there are no equivariant Morse functions.

Example

A connected group would fix the critical points of such a Morse function. Hence, if $G \neq 1$ is connected and the action is free, there no such functions.

As a result, even in finite dimensions standard Morse theory is useless in equivariant (generalized) cohomology calculations.
Hence, we have to go beyond Morse theory.

Morse-Bott functions

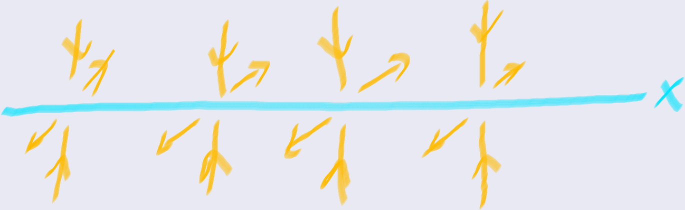
Definition

A *Morse-Bott function* is a smooth function $f : N^n \to \mathbb{R}$ that is locally of the form $f = c \pm x_1^2 \pm \dots x_k^2$ near the critical points (possibly $k < n$).

$crit(f)$ is a disjoint union of submanifolds (locally $x_1 = \cdots = x_k = 0$), called *critical manifolds*.

Example

The critical set and the negative gradient flow of $f = y^2 - z^2$



Morse-Bott functions (cont'd)

There are plenty of equivariant Morse-Bott functions:

Example

Let G act freely on N. Choose a Morse function $f_0 : N/G \to \mathbb{R}$, and let $f : N \to N/G \xrightarrow{f_0} \mathbb{R}$ be the composition. f is Morse-Bott, the critical manifolds are G-torsors.

Example

Let $N = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ and let $G = U(1) = S^1 \subset \mathbb{C}^*$ act by complex multiplication. Then, $f = |z_2|^2 = 1 - |z_1|^2$ is equivariant Morse–Bott. The critical manifolds are two circles defined by $z_1 = 0, |z_2| = 1$ and $|z_1| = 1, z_2 = 0$.

Main results (informal statements)

We package the data of $crit(f)$ and the moduli of gradient trajectories into a topological category called a *framed Morse-Bott flow category*. Our first main result is

Theorem (Côté-K.)

One can recover the (stable) equivariant homotopy type of N from the framed Morse-Bott flow category, i.e. from the critical manifolds and the gradient trajectories between them.

This has corollaries such as equivariant Morse(-Bott) inequalities in extraordinary cohomology.

Main results (informal statements)

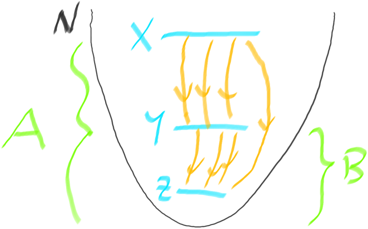
The construction depends on the data of critical points and moduli of gradient trajectories (with some structure): generalizes to Floer theory. Let M be exact symplectic, and consider $SH(M, \mathbb{S})$- a (stable) homotopy type obtained by applying Cohen–Jones–Segal to the free loop space $Map(S^1, M) = \mathcal{L}M$. We prove

Theorem (Côté-K.)

a. *$SH(M, \mathbb{S})$ can be endowed with a S^1-action.*
b. *(Viterbo isomorphism) if $M = T^*Q$, then $SH(M, \mathbb{S})$ is S^1-equivariantly equivalent to $\mathcal{L}Q$ (roughly).*

Recovering the homotopy type from Morse-Bott theory

Constructions in Floer theory should first be explained in Morse theory: How to recover the (stable) homotopy type from Morse–Bott theory? Let $f : N \to \mathbb{R}$ be a Morse-Bott function, $X, Y, Z, \ldots$ critical manifolds, A, B sublevel sets.



Let T^dX denote the descending bundle on X (i.e. the descending directions). Observe, $A/B \simeq X^{T^dX}$, the Thom space of T^dX.

Recovering the homotopy type from Morse-Bott theory (cont'd)

In other words, the difference between the adjacent sublevel sets A and B is given by X^{T^dX}. Similarly, the difference of B with the next sublevel set is Y^{T^dY}, and so on.

Upshot: we can build N from the Thom spaces X^{T^dX}, Y^{T^dY}, Z^{T^dZ}, ...

Remark

Morse–Bott functions also give rise to relative cell decompositions, i.e. one obtains N by gluing disc bundles over the critical manifolds.

Morse-Bott flow categories and framings

Let $\mathcal{M}_f$ denote the (topological) category satisfying

1. $ob(\mathcal{M}_f) = crit(f)$
2. morphisms are given by broken gradient trajectories
3. composition is given by concatenation

For components X, Y of $crit(f)$, let $\mathcal{M}_f(X, Y)$ denote the morphisms from a point of X to a point of Y. Thus, we have

- evaluation maps $\mathcal{M}_f(X, Y) \to X, Y$
- composition $\mathcal{M}_f(X, Y) \times_Y \mathcal{M}_f(Y, Z) \hookrightarrow \partial\mathcal{M}_f(X, Z)$

Morse-Bott flow categories and framings (cont'd)

Let $V_X := T^d X, \ldots$. We have a relative framing on $\mathcal{M}_f(X, Y)$

$$V_X \oplus T_X \simeq T_{\mathcal{M}_f(X,Y)} \oplus \underline{\mathbb{R}} \oplus V_Y$$

Definition (Zhou,Côté-Kartal)

A *framed Morse-Bott flow category* is a (non-unital) topological category $\mathcal{M}$ whose object space is a smooth manifold, whose morphism spaces are smooth manifolds with corners (+further properties).
Moreover, each X, Y are endowed with (virtual) bundles V_X, V_Y, and as part of the data we have the identities

$$V_X \oplus T_X \simeq T_{\mathcal{M}(X,Y)} \oplus \underline{\mathbb{R}} \oplus V_Y$$

Geometric realization

- one can use $X \leftarrow \mathcal{M}(X,Y) \rightarrow Y$ as a correspondence to define $H_*(X) \rightarrow H_*(Y)$
- the identity $V_X \oplus T_X \simeq T_{\mathcal{M}(X,Y)} \oplus \mathbb{R} \oplus V_Y$ allows us to write "correspondence maps" $X^{V_X} \rightarrow \Sigma Y^{V_Y}$

Hence, we obtain a "chain complex" in spaces (spectra) in $X^{V_X}, Y^{V_Y}, \ldots$.

Definition (Côté-K.)

Let $|\mathcal{M}|$ denote the realization of this "chain complex". We call $|\mathcal{M}|$ the *geometric realization* of the framed flow category $\mathcal{M}$.

Example

When there are only two critical manifolds, X, Y, this complex is exactly $X^{V_X} \rightarrow \Sigma Y^{V_Y}$, and one obtains its realization by taking a cone (and suspending).

Main result in differential topology

Theorem (Côté-K.)

$|\mathcal{M}_f|$ is (stable) homotopy equivalent to N. Moreover, if there is action of a compact group G on N, and (f, g) are equivariant, this is a G-equivariant equivalence (as genuine G-spectra).

When the group action is free, one can always lift a Morse-Smale pair on N/G. When the action is not free, one can replace N by $N \times EG$ (which costs us the genuineness of the equivalence though).

Crash course on Hamiltonian Floer theory

Let M be exact symplectic (e.g. an affine variety or cotangent bundle). Given $H : S^1 \times M \to \mathbb{R}$, a time dependent Hamiltonian, there is an associated vector field X_H, the Hamiltonian vector field. For appropriate H and generic (almost) complex structure J on M, one has an action functional $\mathcal{A} : \mathcal{L}M \to \mathbb{R}$ such that

- $\mathcal{A}$ is Morse with critical points given by periodic orbits of X_H
- the gradient trajectories $\mathbb{R} \to \mathcal{L}M$ correspond to holomorphic cylinders $\mathbb{R} \times S^1 \to M$

We denoted the corresponding (stable) homotopy type by $SH(M, \mathbb{S})$.

Equivariant Hamiltonian Floer theory

$\mathcal{A}$ depends on H and J, which cannot be chosen to be S^1-equivariant. Hence, one cannot make the Floer action functional S^1-equivariant, and the action on $\mathcal{L}M$ does not a priori induce an action on $SH(M, \mathbb{S})$. Instead, we imitate the previous construction: $ES^1 = S^\infty$, use an S^1-equivariant Morse–Bott action functional on "$S^\infty \times \mathcal{L}M$":

- this leads to an S^1-equivariant framed Morse–Bott flow category; hence, to an S^1-equivariant homotopy type
- underlying (non-equivariant) homotopy type is $SH(M, \mathbb{S})$

Restriction maps and Viterbo isomorphism

Given (exact, compact) Lagrangian $Q \subset M$, we have:

Theorem (Côté-K.)

There are S^1-equivariant restriction maps $SH(M, \mathbb{S}) \to (\mathcal{L}Q)^{-W_{mas}}$, where W_{mas} is a natural virtual bundle on $\mathcal{L}Q$, and this is an equivalence of (Borel) equivariant spectra when $M = T^*Q$.

We introduce an abstract framework in order to define the maps $|\mathcal{M}_f| \to N^\nu$ (Morse case), as well as $SH(M, \mathbb{S}) \to (\mathcal{L}Q)^{-W_{mas}}$.

Applications I (joint work in progress with Côté):

The symplectic cohomology is not sensitive to the homotopy type of M. However, Albers-Cieliebak-Frauenfelder, Zhao showed how to recover rational homology from the filtered equivariant homotopy type of the symplectic cohomology.
$SH(M, \mathbb{S})$ also has a natural filtration. Then

- one can recover the complex K-theory of M from the filtered, equivariant homotopy type of $SH(M, \mathbb{S})$
- one can also recover Morava K-theories, as well as integral homology of M
- (speculation) presumably, one can also recover even finer information, such as stable homotopy groups from chromatic tower and fracture square

These are both as ungraded vector spaces. The significance of this is in its possible relations to mirror symmetry.

Applications II (joint work in progress with Côté):

Expectation: $SH(M, \mathbb{S})$ is the *topological Hochschild homology* of the (spectral, wrapped) Fukaya category of M. Hence, it carries a *cyclotomic structure*, i.e. an S^1-action, and equivariant Frobenius maps.
We have a proposal for the Frobenius map and checking S^1-equivariance is very easy thanks to the description above.
This is part of a long term project to understand THH (and K-theory of Fukaya categories).

Application III (joint work in progress with Côté and Mak):

Expectation from physics: given Hamiltonian G-action on M, we expect an algebra map $R_*^G(\Omega G) \to SH_G^*(M, R)$, where $R_*^G(\Omega G)$ is the pure Coulomb branch algebra, and R is homology, complex K-theory or elliptic cohomology.

We can construct a G-equivariant model for $SH(M, \mathbb{S})$ using the framework above, and its equivariant R-homology is $SH_G^*(M, R)$. We work on defining the algebra map using parametrized homotopy theory.

Thank you!