Oriented Posets and Rank Matrices

(partially based on joint work with Mohan Ravichandran, Emine Yıldırım and Cem Yalım Özel)

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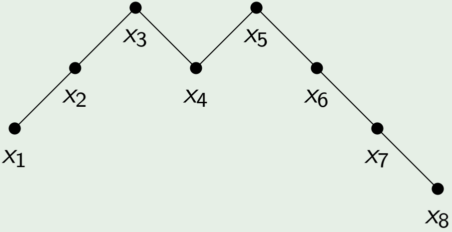
The case of fence posets

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be a composition of n. The *fence poset* of α, denoted $F(\alpha)$ is the poset on $x_1, x_2, \ldots, x_{n+1}$ with the order relations:

$$x_1 \preceq x_2 \preceq \cdots \preceq x_{\alpha_1+1} \succeq x_{\alpha_1+2} \succeq \cdots \succeq x_{\alpha_1+\alpha_2+1} \preceq x_{\alpha_1+\alpha_2+2} \preceq \cdots$$

Example ($\alpha = (2,1,1,3)$)



For a composition of n, we get a poset of $n+1$ nodes.

An *ideal* of a fence is a down-closed subset: $x \in I$, $y \preceq x \Rightarrow y \in I$.

$$\#I = rank(I)$$

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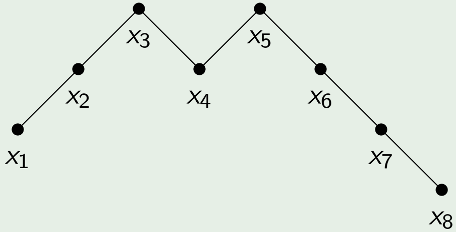

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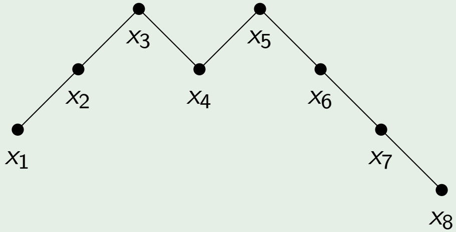

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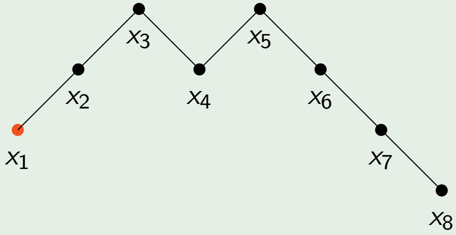

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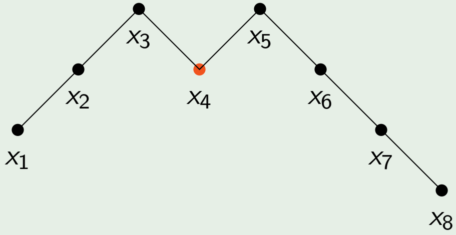

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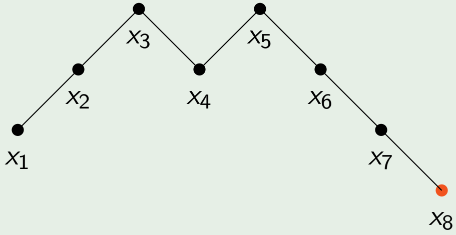

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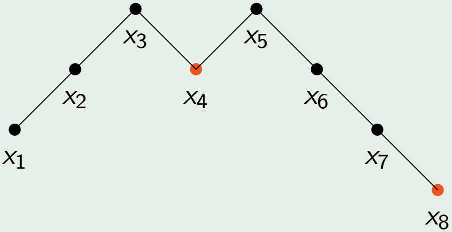

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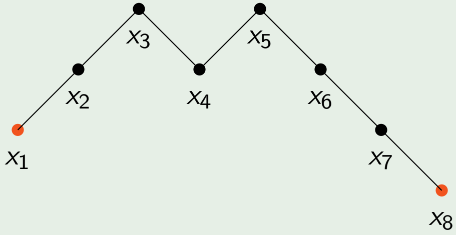

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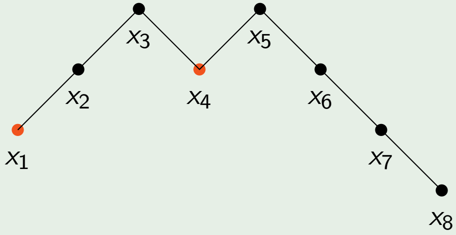

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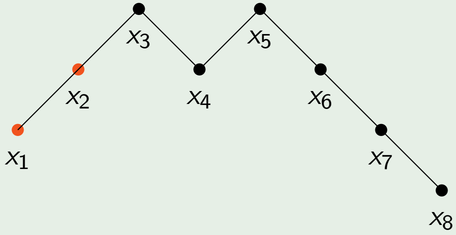

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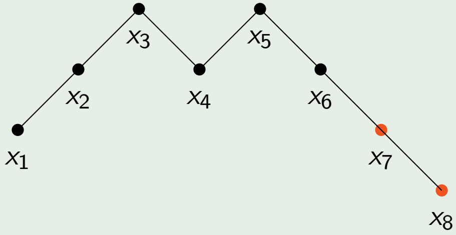

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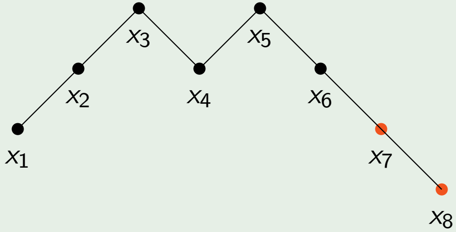

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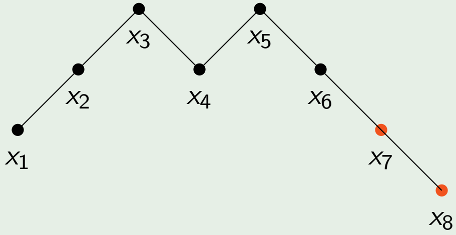

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$1+3q+5q^2+6q^3+6q^4+5q^5+3q^6+2q^7+q^8 \leftarrow$ Rank polynomial.

We can also see ideals of a fence as sub-representations of a quiver representation.

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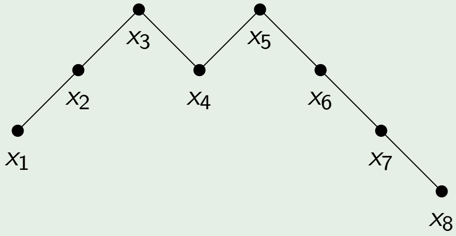

$$\mathbb{k} \longleftarrow \mathbb{k} \longleftarrow \mathbb{k} \longrightarrow \mathbb{k} \longleftarrow \mathbb{k} \longrightarrow \mathbb{k} \longrightarrow \mathbb{k} \longrightarrow \mathbb{k}$$

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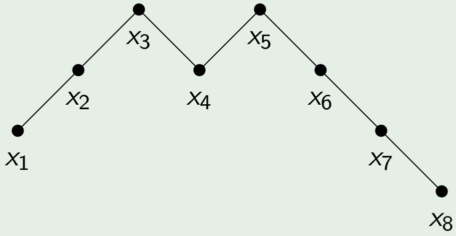

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This is a "type A" quiver representation.

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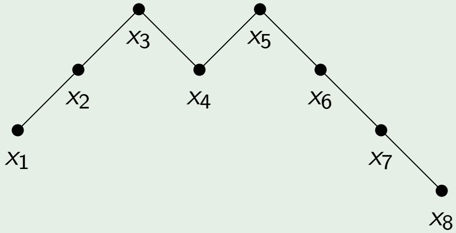

$$\begin{array}{ccccccccccccccc}
\Bbbk & \longleftarrow & \Bbbk & \longleftarrow & \Bbbk & \longrightarrow & \Bbbk & \longleftarrow & \Bbbk & \longrightarrow & \Bbbk & \longrightarrow & \Bbbk & \longrightarrow & \Bbbk \\
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* & \longleftarrow & * & \longleftarrow & * & \longrightarrow & * & \longleftarrow & * & \longrightarrow & * & \longrightarrow & * & \longrightarrow & *
\end{array}$$

A subrepresentation is one that makes the diagram commute.

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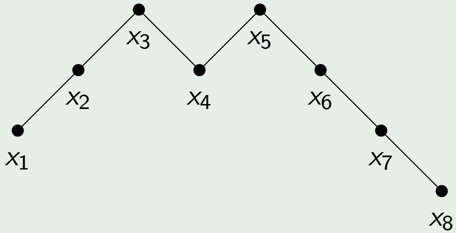

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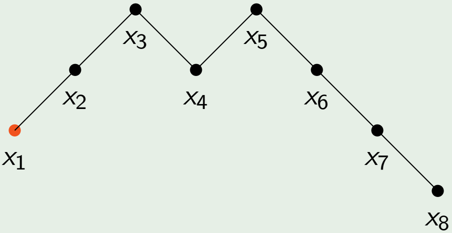

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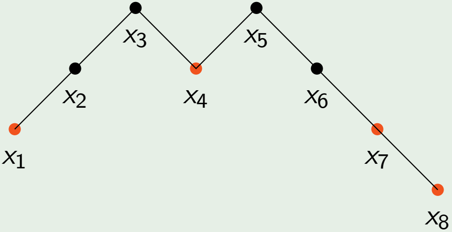

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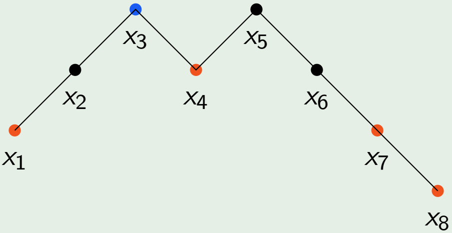

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A q-deformation for rational numbers

Recently, a q-deformation rational numbers was introduced by Morier-Genoud and Ovsienko[1]. Their definition has a *convergence* property, which allows us to extend them to real numbers.

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For a given rational number r/s, we first write it as a continued fraction.

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m}}}}} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cfrac{1}{\ddots - \cfrac{1}{c_k}}}}$$

$$a_i \in \mathbb{Z},\ a_i \geq 1 \text{ for } i \geq 2 \qquad c_i \in \mathbb{Z},\ c_i \geq 2 \text{ for } i \geq 2$$

[1] Morier-Genoud and Ovsienko, "*q*-deformed rationals and *q*-continued fractions".

A q-deformation for rational numbers

Then we replace the expansion terms with q-integers (q^{-1}-integers for a_{2k}), and the 1's with powers of q.

$$\left[\frac{r}{s}\right]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{\ddots + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}} = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{\ddots - \cfrac{q^{c_{k-1}-1}}{[c_k]_q}}}$$

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Another cool thing: $\left[\frac{r}{s}\right]_q = \frac{R(q)}{S(q)}$ where $R(q), S(q) \in \mathbb{Z}[q]$ are polynomials that :ively.

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Another cool thing: $\left[\frac{r}{s}\right]_q = \frac{R(q)}{S(q)}$ where $R(q), S(q) \in \mathbb{Z}[q]$ are polynomials that at q=1, evaluate to r and s respectively.

Also, when $\frac{r}{s} \geq 0$ the coefficients are non-negative.

Example

$$\frac{32}{9} = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}} = 4 - \cfrac{1}{3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2}}}}$$

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$$\left[\frac{32}{9}\right]_q = \frac{1+3q+5q^2+6q^3+6q^4+5q^5+3q^6+2q^7+q^8}{1+2q+2q^2+2q^3+q^4+q^5}.$$

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In general, if r/s corresponds to $[a_1, a_2, \ldots, a_{2m}]$, we have

$$\left[\frac{r}{s}\right]_q = \frac{\text{Rank polynomial for } (a_1 - 1, a_2, a_3, \ldots, a_{2m} - 1)}{\text{Rank polynomial for } (0, a_2 - 1, a_3, \ldots, a_{2m} - 1)}$$

A closer look at rank sequences for fences

$$
\begin{aligned}
(2,1,1,3) &\rightarrow (1,3,5,6,6,5,3,2,1)\\
(3,1,1,2) &\rightarrow (1,2,3,5,6,6,5,3,1)\\
(1,2,1,3) &\rightarrow (1,3,5,6,6,5,4,2,1)\\
(1,1,2,3) &\rightarrow (1,3,5,7,7,5,4,2,1)\\
(2,2,3) &\rightarrow (1,2,4,5,6,6,4,2,1)\\
(2,3,2) &\rightarrow (1,2,4,6,7,6,4,2,1)\\
(2,1,4) &\rightarrow (1,2,3,3,4,4,3,2,1)\\
(2,1,2,1,1) &\rightarrow (1,3,6,7,8,7,5,3,1)
\end{aligned}
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(2,1,4) &\rightarrow (1,2,3,3,4,4,3,2,1)\\
(2,1,2,1,1) &\rightarrow (1,3,6,7,8,7,5,3,1)
\end{aligned}$$

Conjecture (Morier-Genoud, Ovsienko, 2020)

The rank polynomials of fence posets are unimodal.

What more can we say?

Consider $(2,1,1,3) \to (1,3,5,6,6,5,3,2,1)$.

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Consider $(2, 1, 1, 3) \to (1, 3, 5, 6, 6, 5, 3, 2, 1)$.

We have $1 \leq 1 \leq 2 \leq 3 \leq 3 \leq 5 \leq 5 \leq 6 \leq 6$.

We call such a sequence bottom-interlacing:

$$a_n \leq a_0 \leq a_{n-1} \leq a_1 \leq \ldots \leq a_{\lfloor n/2 \rfloor}. \tag{BI}$$

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We call similarly have top-interlacing sequences:

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For example, the rank sequence $(1, 2, 4, 5, 6, 6, 4, 2, 1)$ of $(2, 2, 3)$ is top interlacing:

$$1 \leq 1 \leq 2 \leq 2 \leq 4 \leq 4 \leq 5 \leq 6 \leq 6.$$

What more can we say?

$$
\begin{array}{rcl}
(2,1,1,3) & \rightarrow & (1,3,5,6,6,5,3,2,1) \rightarrow \text{BI} \\
(3,1,1,2) & \rightarrow & (1,3,5,6,6,5,3,2,1) \rightarrow \text{BI} \\
(1,2,1,3) & \rightarrow & (1,3,5,6,6,5,4,2,1) \rightarrow \text{BI} \\
(1,1,2,3) & \rightarrow & (1,3,5,7,7,5,4,2,1) \rightarrow \text{BI} \\
(2,2,3) & \rightarrow & (1,2,4,5,6,6,4,2,1) \rightarrow \text{TI} \\
(2,3,2) & \rightarrow & (1,2,4,6,7,6,4,2,1) \rightarrow \text{BI}, \text{TI} \ \textit{(symmetric)} \\
(2,1,4) & \rightarrow & (1,2,3,3,4,4,3,2,1) \rightarrow \text{TI} \\
(2,1,2,1,1) & \rightarrow & (1,3,6,7,8,7,5,3,1) \rightarrow \text{BI}
\end{array}
$$

What more can we say?

$$(2,1,1,3) \to (1,3,5,6,6,5,3,2,1) \to \text{BI}$$
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Conjecture (McConville, Sagan, Smyth, 2021[2])

Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$.

(a) If $s = 1$ then $r(\alpha) = (1, 1, \ldots, 1)$ is symmetric.

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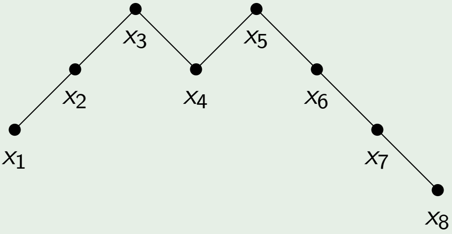
[2] McConville, B. E. Sagan, and Smyth, *On a rank-unimodality conjecture of Morier-Genoud and Ovsienko*.

Idea

What if we close up the fence?

Example ($\alpha = (2,1,1,3)$)



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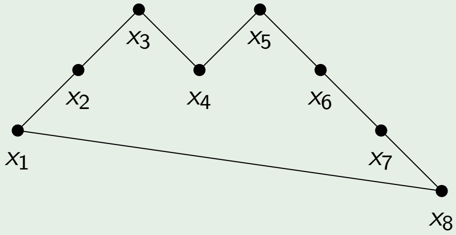

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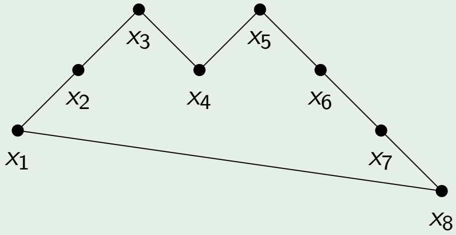

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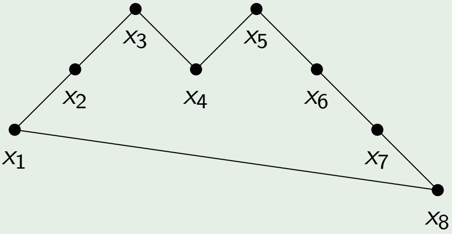

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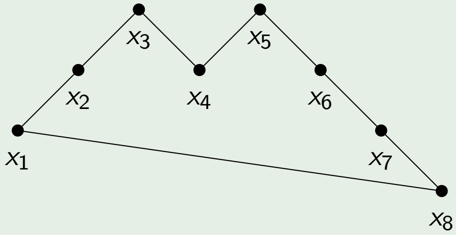

The *circular* fence has rank sequence $(1, 2, 3, 4, 4, 3, 2, 1)$.

It is symmetric. Is this always so?

Answer: Yes, but it is not trivial to prove.

Theorem (Kantarcı Oğuz, Ravichandran, 2021[3])

Rank polynomials of circular fence posets are symmetric.

[3] Kantarcı Oğuz and Ravichandran, *Rank Polynomials of Fence Posets are Unimodal*.

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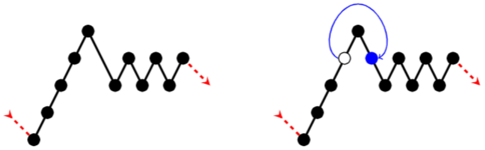
We have one case that is trivially symmetric: $(k, 1, 1, \ldots, 1)$.



We show that moving a node from one segment to the next does not break symmetry.

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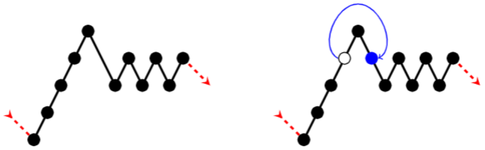

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$>>$ Recent bijective proof by Sagan and Elizalde[4].

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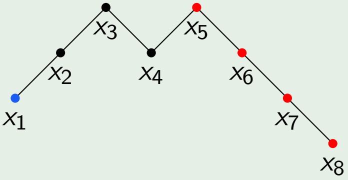
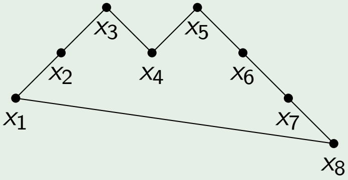
The next step

There are several natural ways to associate a circular fence to a given fence.

Example (Adding the relation $x_1 \succeq x_8$)





$$\sum_{I} q^{\text{rank}(I)} = \sum_{\{I \mid x_1 \in I \Rightarrow x_8 \in I\}} q^{\text{rank}(I)} + \sum_{\{I \mid x_1 \in I, x_8 \notin I\}} q^{\text{rank}(I)}$$

circular rank polynomial *(symmetric)*

$q \times$ rank polynomial for $(1, 1)$ *(smaller, shifted center)*

What does this tell us about the rank polynomial?

symmetric piece	$(1,2,3,5,5,5,3,2,1)$	$b_0 = b_n,\ b_1 = b_{n-1}, \dots$
$+$	$+$	
smaller piece, shifted center	$(0,1,2,1,1,0,0,0,0)$	$c_0 \geq c_n,\ c_1 \geq c_{n-1}, \dots$
$=$	$=$	
$\sum_I q^{\text{rank}(I)}$	$(1,3,5,6,6,5,3,2,1)$	$a_0 \geq a_n,\ a_1 \geq a_{n-1}, \dots$

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This gives us half of the equations for being bottom interlacing:

$$a_n \leq a_0, \quad a_{n-1} \leq a_1, \quad a_{n-2} \leq a_2 \quad a_{n-3} \leq a_3, \ldots$$

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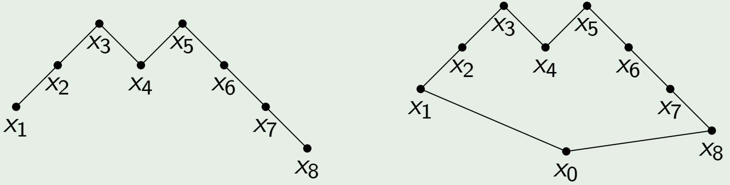
$$a_n \leq a_0 \leq a_{n-1} \leq a_1 \leq a_{n-2} \leq a_2 \leq a_{n-3} \leq a_3 \leq \dots \qquad \text{(BI)}$$

We can get the other half by associating another circular fence.

Example (Connecting x_8 and x_1 by a minimal node x_0)



$$\sum_{\{I \mid x_0 \in I\}} q^{\text{rank}(I)} = \sum_{I} q^{\text{rank}(I)} - \sum_{\{I \mid x_0 \notin I\}} q^{\text{rank}(I)}$$

$q \times$ rank polynomial for $(2,1,1,3)$	circular rank polynomial *(symmetric, shifted center)*	rank polynomial for (0) *(smaller, shifted center)*

On the rank polynomial side

symmetric piece larger	$(1,2,3,5,6,6,5,3,2,1)$	$b_0 = b_{n+1},\ b_1 = b_n, \ldots$
$-$	$-$	
smaller piece, shifted center	$(1,1,0,0,0,0,0,0,0)$	$c_0 \geq c_n,\ c_1 \geq c_{n-1}, \ldots$
$=$	$=$	
$(0, a_0, a_1, \ldots, a_n)$	$(0,1,3,5,6,6,5,3,2,1)$	$0 \leq a_n,\ a_0 \leq a_{n-1} \ \ldots$

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This gives us the other half of the bottom-interlacing equations:

$$a_n \leq a_0, \quad a_{n-1} \leq a_1, \quad a_{n-2} \leq a_2, \quad a_{n-3} \leq a_3, \ldots$$

$$+$$

$$a_0 \leq a_{n-1}, \quad a_1 \leq a_{n-2}, \quad a_2 \leq a_{n-3}, \ \ldots$$

$$=$$

$$a_n \leq a_0 \leq a_{n-1} \leq a_1 \leq a_{n-2} \leq a_2 \leq a_{n-3} \leq a_3 \leq \ldots \qquad \text{(BI)}$$

Theorem (Kantarcı Oğuz, Ravichandran, 2021)

Rank polynomials of fence posets are unimodal.

In particular, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ we have:

(a) If $s = 1$ then $r(\alpha) = (1, 1, \ldots, 1)$ is symmetric.

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Are they also unimodal? **Answer:** Not always.

For the circular poset $(1, a, 1, a)$ we get a small dip in the middle:

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Conjecture (Kantarcı Oğuz, Ravichandran, 2022)

For any $\alpha \neq (1, k, 1, k)$ or $(k, 1, k, 1)$ for some k, the rank sequence $\overline{\mathcal{R}}(\alpha; q)$ is unimodal.

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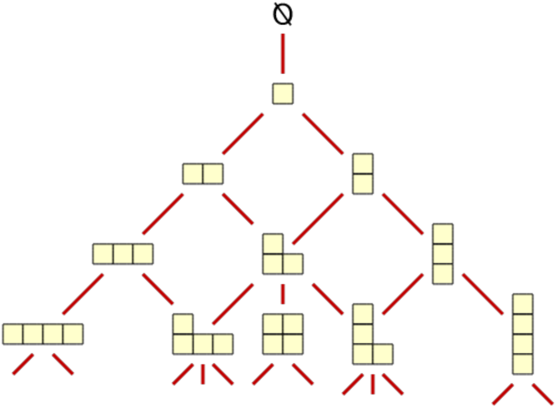
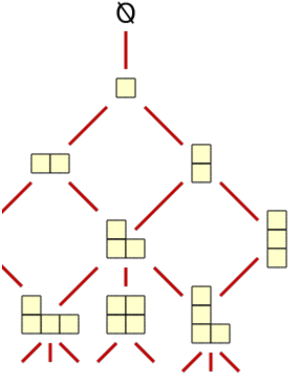
We can also see fences as intervals in the Young's lattice.

Young's Lattice is the lattice of Ferrers diagrams of Partitions ordered by inclusion.



(Image from Wikipedia, created by David Eppstein)

For any partition, we can look at the generating function of the partitions that lay under it.



$$G(\lambda; q) := \sum_{\mu \subset \lambda} q^{|\mu|}$$

$$G\left(\ydiagram{1,2}; q\right) = q^3 + 2q^2 + q + 1$$

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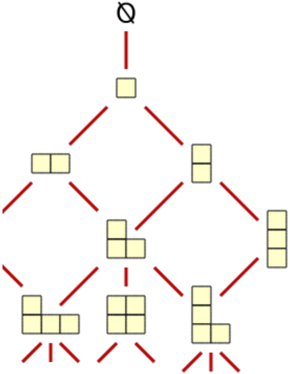

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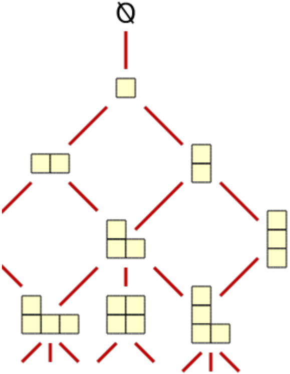

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We can also look at the interval between two partitions.

$$G(\lambda/\nu; q) := \sum_{\nu \subset \mu \subset \lambda} q^{|\mu| - |\nu|}$$

$$G\left((2,1,1)/(1,1); q\right) = q^2 + 2q + 1$$

Unimodality of these polynomials were considered by Stanton in 1990[5].

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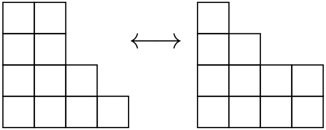

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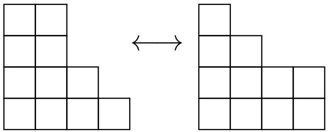

Conjecture (Stanton,1990)

The polynomials corresponding to self-dual partitions are unimodal.

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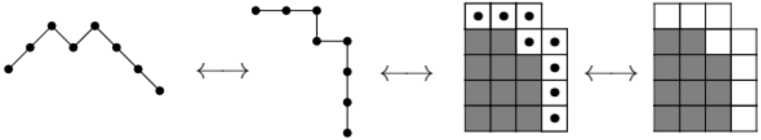
TABLE I

Partition	i	Values	Partition	i	Values
8 8 4 4	15	31 30 31	11 11 6 6	21	67 66 67
10 9 4 4	17	46 45 46	14 13 4 4	21	76 75 76
10 10 4 4	17	46 45 46	16 12 4 4	23	91 90 91
12 10 4 4	19	61 60 61	14 14 4 4	21	76 75 76
12 11 4 4	19	61 60 61	12 12 8 4	23	81 80 81
12 12 4 4	19	61 60 61	12 10 8 6	23	82 81 82
14 11 4 4	21	76 75 76	8 8 8 6 4 2	23	141 140 141
11 11 6 5	21	67 66 67	8 8 6 6 4 4	23	144 143 144
14 12 4 4	21	76 75 76			

(Table from "Unimodality and Young's Lattice", Stanton)

Given a fence, we can see it as a difference of two partitions α/ν.

Example $((2,1,1,3) \to (4,4,4,4,3)/(3,3,3,2))$



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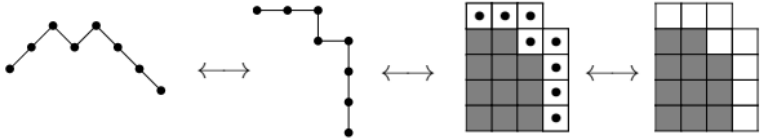

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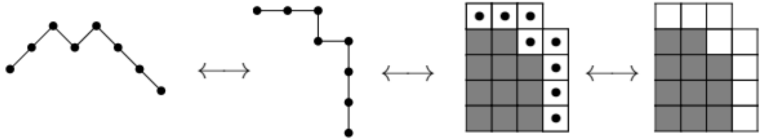

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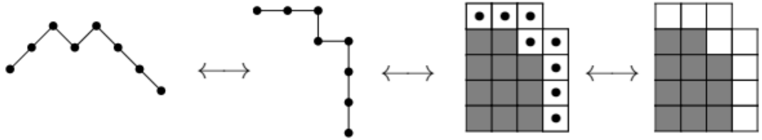

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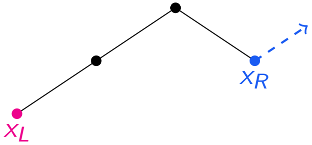
Polynomials corresponding to ribbon diagrams are unimodal.

A generalization: Oriented Posets

We build posets from building blocks which we call *oriented posets*, which come with 2×2 rank matrices instead of rank polynomials.



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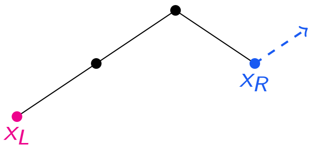

$$\mathcal{M}_q(\mathbf{P} \nearrow) := \begin{bmatrix} q + q^2 + q^3 + q^4 & 1 + q + q^2 \\ q & 1 \end{bmatrix}$$

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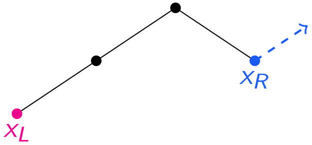
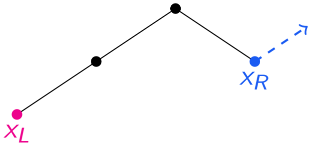


$$\mathcal{M}_q(\mathbf{P} \nearrow) := \begin{bmatrix} q + q^2 + q^3 + q^4 & 1 + q + q^2 \\ q & 1 \end{bmatrix}$$

We can read the rank polynomial directly from the rank matrix.

We build posets from building blocks which we call *oriented posets*, which come with 2×2 rank matrices instead of rank polynomials.



$$\mathcal{M}_q(\mathbf{P}\nearrow) := \begin{bmatrix} q+q^2+q^3+q^4 & 1+q+q^2 \\ q & 1 \end{bmatrix}$$

We can read the rank polynomial directly from the rank matrix.

Rank Matrices

Combining posets $\Leftrightarrow$ Multiplying rank matrices.

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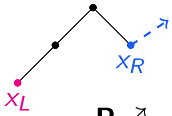
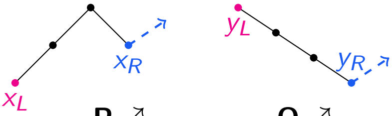


$\mathbf{P}\nearrow$

$\mathcal{M}_q(\mathbf{P}\nearrow)$

Rank Matrices

Combining posets ⇔ Multiplying rank matrices.



$\mathbf{P} \nearrow$ $\qquad$ $\mathbf{Q} \nearrow$

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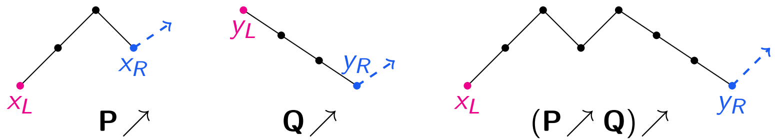
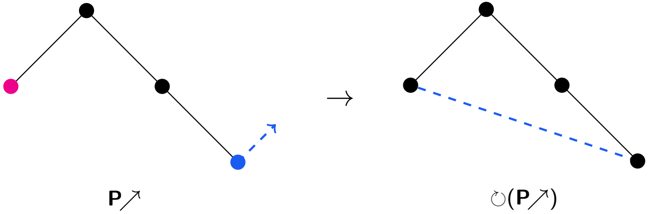


$$\mathcal{M}_q(\mathbf{P}\nearrow) \qquad \mathcal{M}_q(\mathbf{Q}\nearrow) \quad \Rightarrow \quad \mathcal{M}_q((\mathbf{P}\nearrow\mathbf{Q})\nearrow)$$

Taking the trace $\Leftrightarrow$ Combining the two ends of a poset

Rank Matrices

Taking the trace ⇔ Combining the two ends of a poset



$$\mathcal{R}(\circlearrowright(\mathbf{P}\nearrow); q) = \mathrm{tr}(\mathcal{M}_w(\mathbf{P}\nearrow))$$

In particular, for dealing with fence poset or circular fence posets, two matrices are enough to give us all the structure.

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$$\mathcal{M}_w(\bullet\searrow) := D = \begin{bmatrix} 1+q & -q \\ 1 & 0 \end{bmatrix}, \qquad \mathcal{M}_w(\bullet\nearrow) := U = \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix}.$$

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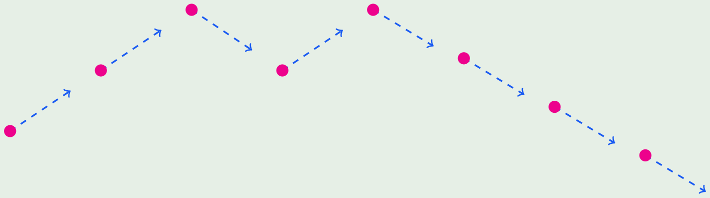

$$\mathcal{M}_q(F(2,1,1,3)\searrow) = U \cdot U \cdot D \cdot U \cdot D \cdot D \cdot D \cdot D.$$

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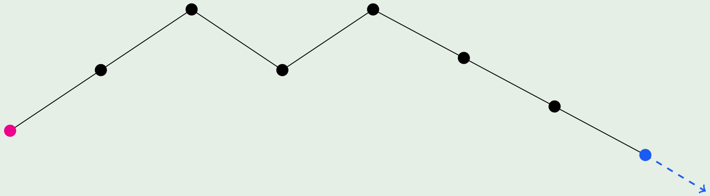

$$\mathcal{M}_q(F(2,1,1,3)\searrow) = U^2 \cdot D \cdot U \cdot D^4.$$

The case of fences

Theorem (Kantarcı Oğuz, 2022)

Consider the oriented poset $F(\alpha)$ corresponding to $\alpha = (u_1, d_1, u_2, d_2, \ldots, u_s, d_s)$.
Then $F(\alpha)$ has rank matrices:

$$\mathcal{M}_q(F(\alpha)\searrow) = U^{u_1} D^{d_1} U^{u_2} D^{d_2} \cdots U^{u_{s-1}} D^{d_{s-1}} U^{u_s} D^{d_s+1},$$
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The circular fence poset $\overline{F}(\alpha)$ has rank polynomial:

$$\mathcal{R}(\overline{F}(\alpha); q) = \mathrm{trace}(U^{u_1} D^{d_1} U^{u_2} D^{d_2} \cdots U^{u_{s-1}} D^{d_{s-1}} U^{u_s} D^{d_s}).$$

Application: Identities

We can use matrices to do fast calculations, conjecture and prove identities.

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Proposition (Kantarcı Oğuz, 2022)

Let $\mathbf{X}$ be a palindromic composition with an even number of parts. For $k \geq 1$, $s \geq 1$ we have:

$$\overline{\mathcal{R}}((1, k, r+1, \mathbf{X}, r); q) = [k+1]_q \cdot \overline{\mathcal{R}}((r+2, \mathbf{X}, r); q),$$
$$\overline{\mathcal{R}}((k, 1, k+r, \mathbf{X}, r); q) = [k+1]_q \cdot \overline{\mathcal{R}}((k+r+1, \mathbf{X}, r); q).$$

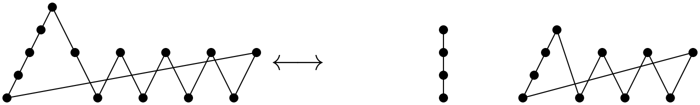

Illustration of (Id 1) with $r = 1$, $k = 4$,$s = 2$.

Application: Recurrences

We can use matrix identities to get recurrences on fences.

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Proposition (Kantarcı Oğuz, Ravichandran, Özel, 2023)

We have the following recurrence relations on rank polynomials:

$$\mathcal{R}((k+2,\mathbf{X});q) = (q+1)\mathcal{R}((k+1,\mathbf{X});q) + q\mathcal{R}((k,\mathbf{X});q),$$
$$\overline{\mathcal{R}}((k+2,\mathbf{X});q) = (q+1)\overline{\mathcal{R}}((k+1,\mathbf{X});q) + q\overline{\mathcal{R}}((k,\mathbf{X});q).$$

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Proposition (Kantarcı Oğuz, Özel, Ravichandran, 2022)

We have the following recurrence relation polynomials:

$$\begin{aligned}\overline{\mathcal{R}}((a,1,b,X);q) =& \overline{\mathcal{R}}((a-1,1,b,X);q) + \overline{\mathcal{R}}((a,1,b-1,X);q) \\ &-\overline{\mathcal{R}}((a-1,1,b-1,X);q) \\ &+\overline{\mathcal{R}}((a+b+1,X);q) - \overline{\mathcal{R}}((a+b,X);q).\end{aligned}$$

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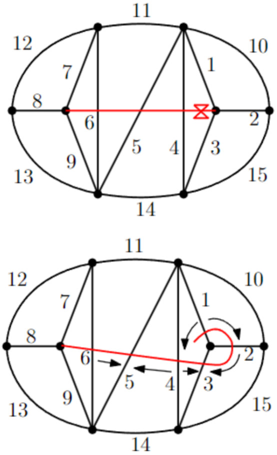
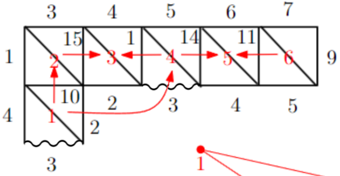
Theorem (Kantarcı Oğuz, Özel, Ravichandran, 2022)

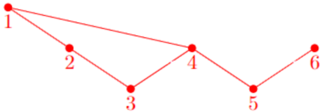
For any $\alpha \neq (1,k,1,k)$ or $(k,1,k,1)$ for some k, the rank sequence $\overline{\mathcal{R}}(\alpha;q)$ is unimodal.

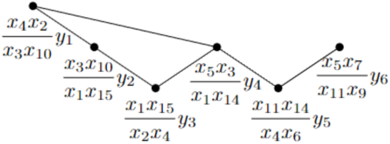
Application: Calculations on Cluster Algebras

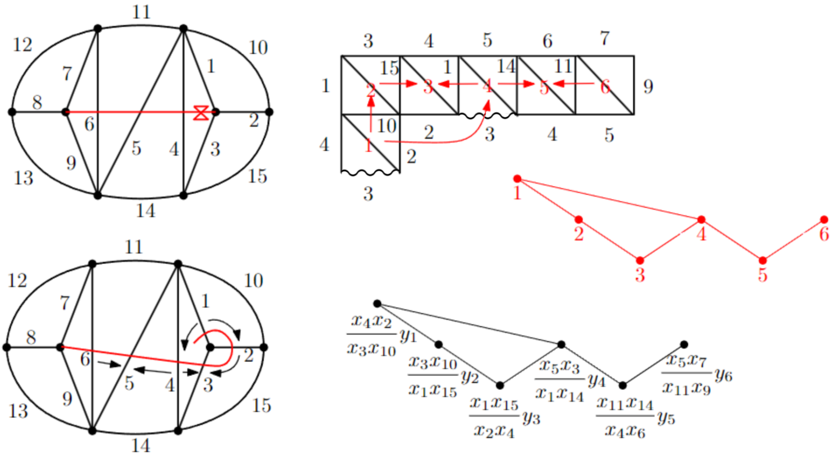
We can also keep track of the actual vertices in each ideal. We only need to substitute w_i for q in the matrices. We can use that to calculate expansion formulas for arcs in trianglated surfaces.











We get a *weight* matrix where the top left entry gives us the generating polynomials of the ideals.

$$\circlearrowleft\nearrow \left(\begin{bmatrix} 1+w_1 & -w_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1+w_2 & -w_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_4 & 1 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} w_5 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1+w_6 & -w_6 \\ 1 & 0 \end{bmatrix}.$$

$$1 + w_3 + w_5 + w_2w_3 + w_3w_5 + w_5w_6 + w_2w_3w_5 + w_3w_4w_5 + w_3w_5w_6 + w_2w_3w_4w_5 + w_2w_3w_5w_6 + w_3w_4w_5w_6 + w_1w_2w_3w_4w_5 + w_2w_3w_4w_5w_6 + w_1w_2w_3w_4w_5w_6.$$

We than plug in the weights (and more) to obtain the expansion formula of the arc:

$$x_\gamma = \frac{x(M_-)}{\mathrm{cross}(\gamma, T)}\mathcal{R}(P_\gamma; xy) = \frac{x_1x_2x_4^2x_6x_9}{x_1x_2x_3x_4x_5x_6}\mathcal{R}(P_\gamma; xy)$$

$$= \frac{x_4x_9}{x_3x_5} + \frac{x_1x_9x_{15}}{x_2x_3x_5}y_3 + \frac{x_9x_{11}x_{14}}{x_3x_5x_6}y_5 + \frac{x_9x_{10}}{x_2x_5}y_2y_3 + \frac{x_1x_9x_{11}x_{14}x_{15}}{x_2x_3x_4x_5x_6}y_3y_5$$
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$$+\frac{x_9x_{11}}{x_1x_6}y_1y_2y_3y_4y_5 + \frac{x_3x_5x_7x_{10}}{x_1x_2x_4x_6}y_2y_3y_4y_5y_6 + \frac{x_5x_7}{x_1x_6}y_1y_2y_3y_4y_5y_6.$$

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All solution triples can be recursively calculated recursively from

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Recently, q-deformed Markov numbers were defined using q-deformed Cohn matrices and dividing by $[3]_q$ instead.

Observations:

The q-deformed Cohn matrices are rank matrices of certain posets.

The division by $[3]_q$ can be dealt with via the identity:

$$\overline{\mathcal{R}}((1, k, r+1, \mathbf{X}, r); q) = [k+1]_q \cdot \overline{\mathcal{R}}((r+2, \mathbf{X}, r); q).$$

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Algorithm: For a given Markov Number N,

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Delete leftmost and rightmost letters of w.

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For the Markov number 13 we get:

$$13 \to aab \to a \to 1, 1 \to (3, 1, 1, 1) = \alpha(13)$$

. The q-deformations given by $\mathcal{R}(\overline{F}(3, 1, 1, 1); q)$:

$$\text{trace}(U^3 \cdot D \cdot U \cdot D) = 1 + 2q + 2q^2 + 3q^3 + 2q^4 + 2q^5 + 1.$$

Thank you for listening!

- Kantarcı Oğuz, E. & Ravichandran, M. Rank Polynomials of Fence Posets are Unimodal. *Discrete Math.*. **346** (2023).
- Kantarcı Oğuz, E. Oriented Posets. (2022).
- Morier-Genoud, S. & Ovsienko, V. q-deformed rationals and q-continued fractions. *Forum Math. Sigma*. **8** pp. Paper No. e13, 55 (2020).
- McConville, T., Sagan, B. & Smyth, C. On a rank-unimodality conjecture of Morier-Genoud and Ovsienko. *Discrete Math.*. **344** pp. 13 (2021).
- Elizalde, S. & Sagan, B. Partial rank symmetry of distributive lattices for fences. (2022).
- Kantarcı Oğuz, E. & Yıldırım, E. Cluster Algebras and Oriented Posets. (2022).
- Kantarcı Oğuz, E. & Özel, C. Y.& Ravichandran, M. Fence Posets and Ehrhart-Equivalence. (2022).

Extra: Rank Matrix Calculations

$$\mathcal{M}_q(\mathbf{P}\searrow) := \begin{bmatrix} \mathcal{R} & -\mathcal{R}|_{x_R \in I} \\ \mathcal{R}|_{x_L \notin I} & -\mathcal{R}|_{\substack{x_R \in I \\ x_L \notin I}} \end{bmatrix} \quad \mathcal{M}_q(\mathbf{P}\nearrow) := \begin{bmatrix} \mathcal{R}|_{x_R \in I} & \mathcal{R}|_{x_R \notin I} \\ \mathcal{R}|_{\substack{x_R \in I \\ x_L \notin I}} & \mathcal{R}|_{\substack{x_R \notin I \\ x_L \notin I}} \end{bmatrix}$$

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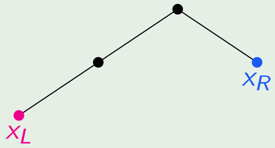
Example



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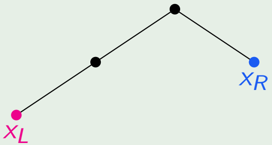

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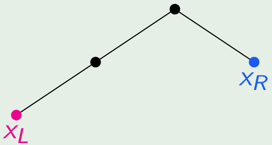

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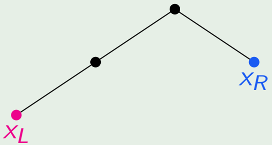

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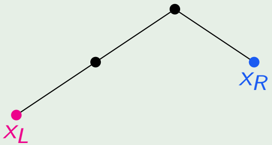
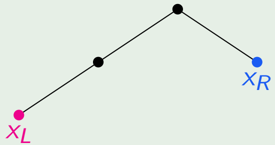
Example



$$\mathcal{M}_q(\mathbf{P}\searrow) := \begin{bmatrix} 1+2q+2q^2+q^3+q^4 & -q-q^2-q^3-q^4 \\ 1+q & -q \end{bmatrix}.$$

$$\mathcal{M}_q(\mathbf{P}\searrow) := \begin{bmatrix} \mathcal{R} & -\mathcal{R}|_{x_R\in I} \\ \mathcal{R}|_{x_L\notin I} & -\mathcal{R}|_{\substack{x_R\in I\\ x_L\notin I}} \end{bmatrix} \quad \mathcal{M}_q(\mathbf{P}\nearrow) := \begin{bmatrix} \mathcal{R}|_{x_R\in I} & \mathcal{R}|_{x_R\notin I} \\ \mathcal{R}|_{\substack{x_R\in I\\ x_L\notin I}} & \mathcal{R}|_{\substack{x_R\notin I\\ x_L\notin I}} \end{bmatrix}$$

Example



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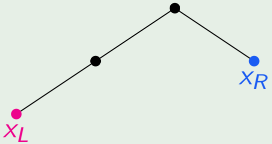

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Extra: Rank Matrix Calculations

$$\mathcal{M}_q(\mathbf{P}\searrow) := \begin{bmatrix} \mathcal{R} & -\mathcal{R}|_{x_R \in I} \\ \mathcal{R}|_{x_L \notin I} & -\mathcal{R}|_{\substack{x_R \in I \\ x_L \notin I}} \end{bmatrix} \quad \mathcal{M}_q(\mathbf{P}\nearrow) := \begin{bmatrix} \mathcal{R}|_{x_R \in I} & \mathcal{R}|_{x_R \notin I} \\ \mathcal{R}|_{\substack{x_R \in I \\ x_L \notin I}} & \mathcal{R}|_{\substack{x_R \notin I \\ x_L \notin I}} \end{bmatrix}$$

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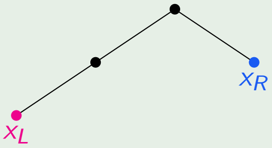

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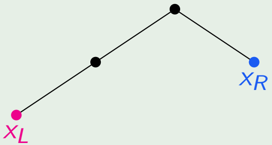

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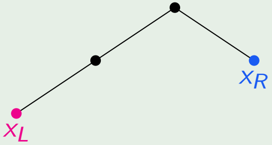

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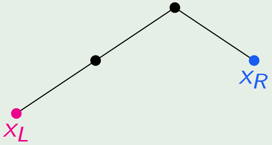

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