Some open problems in nonlinear PDE's

December 12, 2023

Navier - Stokes equations

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{g},$$

$$\nabla\cdot\mathbf{v} = 0,$$

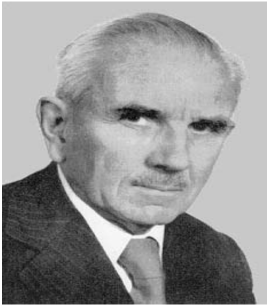
$$\mathbf{v}(x,t) := \mathbf{v}(x_1,x_2,x_3,t) = \begin{bmatrix} v_1(x_1,x_2,x_3,t) \\ v_2(x_1,x_2,x_3,t) \\ v_3(x_1,x_2,x_3,t) \end{bmatrix}$$

$$\Delta\mathbf{v} := \begin{bmatrix} \Delta v_1(x,t) \\ \Delta v_2(x,t) \\ \Delta v_3(x,t) \end{bmatrix}, \quad \nabla\cdot\mathbf{v} := \sum_{k=1}^{3} \frac{\partial v_k}{\partial x_k}$$

$$\nabla p := \begin{bmatrix} \frac{\partial p}{\partial x_1} \\ \frac{\partial p}{\partial x_2} \\ \frac{\partial p}{\partial x_3} \end{bmatrix},$$

"Fluid dynamicists were divided into hydraulic engineers who observe what cannot be explained and mathematicians who explain things that cannot be observed"

. Cyril Hinshelwood,



Jean Leray (1906- 1998)

Weak solution.(Leray-Hopf çözümü) A function $\mathbf{v} \in L^\infty([0,T],H) \cap L^2([0,T],V)$ that satisfies

$$\int_0^T [(\mathbf{v},\eta_t) + \nu(\nabla \mathbf{v}, \nabla \eta) + ((\mathbf{v}\cdot\nabla)\mathbf{v},\eta)]\, dt = (\mathbf{u}_0,\eta)$$

for each $\eta \in C_0^\infty([0,T],\mathbb{R}^n)$.

J. Leray, (1934). Essai sur les Mouvements Plans d'un Liquide Visqueux que Limitent des Parois, J. Math. Pures Appl., **13**, 331-418.
J. Leray, (1934) Sur le mouvement d'un liquide visqueux emplissant l'espace. (French) Acta Math. **63** no. 1, 193–248.

Eberhard Hopf (1901-1983)

E. Hopf, 1950/1951, Über die Anfganswertaufgabe für die Hydrodynamischen Grundgleichungen, Math. Nachr. **4** (1951), 213-231.

Jean Leray.

- $n = 2$: The Cauchy problem for Navier-Stokes (NS) equations has a global solution (i.e. for all $T > 0$) and the solution is unique. If the nitial function is smooth enough then the solution is smooth enoug.
- $n = 3$: There exists T_0 such that the Cauchy problem for the NS has a unique smooth solution on $[0, T_0)$.
- The problem has a weak solution that is equal to smooth one on $\mathbb{R}^3 \times [0, T_0)$.
- If $[0, T_0)$ is the maximal interval of existence, then for each $p > 3$ ithere exists $\epsilon_p > 0$ such that

$$t \to T_0 \quad \text{iken} \quad \left(\int_{\mathbb{R}^3} |\mathbf{v}|^p dx \right)^{\frac{1}{p}} \geq \frac{\epsilon_p}{(T_0 - t)^{\frac{1}{2}(1 - \frac{3}{p})}}$$

O.A. Ladyzhenskaya (1922-2004)

O.A. Ladyzhenskaya The Mathematical Theory of Viscous Incompressible Flow 1961, Moskova.

Giovanni Prodi
(1925–2010)

James Serrin
(1926-2012)

Ladyzhenskaya - Prodi -Serrin condition.

When $n = 3$ the Cauchy problem for NS equations (or in.bvp)) has not two different solutions belonging to

$$L^{\ell}(0; T; L^{s}(\Omega)), \quad \frac{2}{\ell} + \frac{3}{s} = 1, \quad s > 3$$

To prove uniueness of the weak solution we need the estimate

$$\int_0^T \|v(t)\|^{\ell}_{L^s(\Omega)} dt \leq C, \quad \forall T > 0$$

Moeover if the intial function is smooth enogh the corresponding solution is also smooth enough.

L. Iscauriaza, G. Seregin, V. Shverak proved that it suffices to have the following estimate

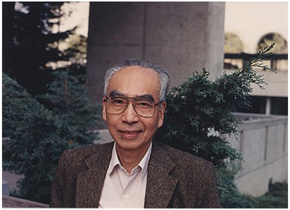
$$\int_{\Omega} |v(x,t)|^3 dx \leq C, \quad \forall t \in \mathbb{R}^+$$

to prove uniqueness of a weak solution.

L. Iscauriaza, G. A. Seregin and V. Shverak, Russian Math. Surveys **58** (2003) 211-250.

Smoothness of solutions of the probel with initial data: $\mathbf{v}_0(x, y, z, t) = (\phi_r^0(r, z), \phi_\theta^0(r, z), \phi_z^0(r, z))$

[1] O. Ladyzhenskaya , Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova **7** (1968)
[2] M. Ukhovskii and V. Yudovich, J. Appl. Math. Mech. **32** (1968),



T.Kato (1925–2010)

V.I.Yudovich (1926-2012)

H. Fujita and T. Kato proved regularity of solution of the Cauchy problem for NS equations with small initial data.

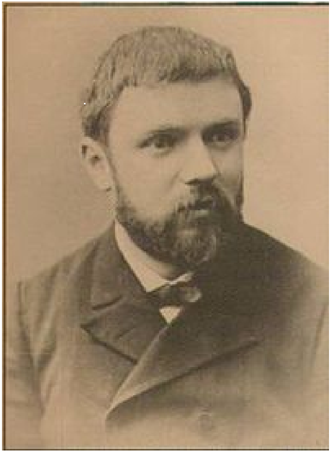
H. Fujita, T. Kato, On the Navier-Stokes initial value problem. I, Arch. Rational Mech. Anal. **16** (1964), 269-315.

C.L Fefferman, (2006) Existence and Smoothness of the Navier-Stokes Equation. The Millennium Prize Problems, Clay Mathematics Institute, Cambridge, 57-67.

Existence and smoothness of a global solution to 3D Navier-Stokes solutions.

Mathematics of 20 century

David Hilbert
(1862–1943)



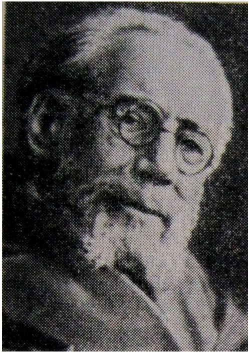
Henry Poincare
(1854-1912)

The Mathematical Problems of David Hilbert

Problem 19: "Are the solutions of regular problems in the calculus of variations always necessarily analytic?"
Problem 20 The general problem of boundary values. (Variational problems.)
Problem 23 Further development of the methods of the calculus of variations.

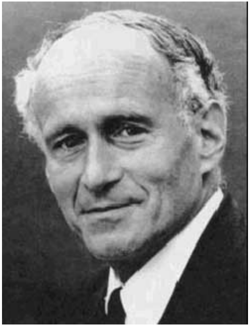
Ennio de Giorgy
(1928–1996)

John Nash (1928-2015)



N. M. Günter
S.L.Sobolev'in doktora tezi danışmanı

S. L. Sobolev



L. Schwartz

Ciprian Foias (1933 - 2020)

- C. Foias , G. Prodi , Sur le comportement global des solutionsdes équations de Navier-Stokes en dimension 2, Rend. Sem. Mat. Univ. Padova **39** (1967),1-47.
- O. A. Ladyzhenskaya, On the dynamical generated by the Navier-Stokes equatons , Zap. nauchn. senin LOMI, Leningrad **27**(1972)

O.A. Ladyzhenskaya

Determining modes, feedback control

Reaction - diffusion equation

$$u_t - u_{xx} - \lambda u + u^3 = h$$

Damped nonlinear wave equation

$$u_{tt} - u_{xx} + bu_t - \lambda u + u^3 = h$$

Cahn - Hilliard equation

$$u_t + \Delta^2 u + \Delta(\lambda u - u^3) = h$$

Phase field equations

$$\begin{cases} \tau\phi_t = \xi^2\Delta\phi - g(\phi) + 2u + h, \\ u_t + \frac{\ell}{2}\phi_t = K\Delta u \end{cases}$$

What is the minimal number of modes that determines the asymptotic behavior of solutions as $t \to \infty$?

Sergey Bernstein (1980-1968)

S. Bernstein, *On a class of functional differential equations.* Izv. Aad. Nauk SSSR, 1940, 17-26.

S.N. Bernstein proved that if the functions u_0 and u_1 are real analytic, then the problem

$$\begin{cases} u_{tt}(x,t) - \left(1 + \int_0^L u_x^2(x,t)dx\right) u_{xx}(x,t) = 0, \ \ x \in (0;L), t > 0, \\ u(0,t) = u(L,t) = 0, \ \ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \ \ x \in (0,L) \end{cases}$$

has a global solution that is real analytic.

Has the problem (K) a global solution for initial data that have only finitely many derivatives?

Jacques-Louis Lions (1928-2001)

J.-L. Lions *Quelques methods de resolution des problems aux limites non lineaires.* Dunod Gauthier-Villars, 1969

$$u_{tt} - \Delta u + u_t^3 = 0, \quad x \in \Omega, t > 0. \qquad (T)$$

Are solutions of the initial boundary value problem for (T) sufficiently smooth, if the initial data are sufficiently smooth?

Vladimir Zakharov (1939-2023)

Zakharov system

$$\begin{cases} u_{tt} - \Delta u = \Delta(|\psi|^2), \\ \psi_t + \Delta\psi = \psi u \end{cases}$$

$$u_t - 6uu_x + u_{xxx} = 0$$

$$u_t - 6u^m u_x + u_{xxx} = 0, \quad m = 6, 7, \cdots\cdots \qquad (A)$$

Solutions of the Cauchy problem for the generalized KDV equation (A) that blow up in a finite time.

Y. Martel and F. Merle , Stability of blow-up profile and lowerbounds for blow-up rate for the critical generalized KdV equation, Annals of Mathematics,155(2002),235–280

C. Gardner, J. Green, M.Kruskal, R. Miura
Phus. Rev. Letters (1967)

$$u_t - 6uu_x + u_{xxx} = 0$$

Conservation laws

$$\int_{\mathbb{R}} u(x,t)dx = C, \ \int_{\mathbb{R}} u^2(x,t)dx = C,$$

$$\int_{\mathbb{R}} \left[u^3(x,t) - \frac{1}{2}u_x^2(x,t) \right] dx = C, \cdots$$

$$\int_{\mathbb{R}}, \left[u^{2k+1}(x,t) + \cdots \right] dx = C, \quad \forall t \in \mathbb{R}$$

R. Miura, SIAM Rev. (1976)

$$u_{tt} - u_{xx} = \sin(u), \quad v_{tx} = \sin(u)$$

$$u_{tt} - u_{xx} = e^{u}$$

$$u_{tt} - u_x^2 u_{xx} = 0$$

$$u_{tt} - e^{u_x} u_{xx} = 0$$

Simple algorithms for deriving infinitely many conservation laws ???

Kuramoto - Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} + u_x^2 = 0$$

2D Kuramoto - Sivashinsky equatin

$$u_t + \Delta u + \Delta^2 u + |\nabla u|^2 = 0, \;\; x \in \mathbb{R}^2, t > 0.$$

Global well-posedness for arbitrary smooth initial data.

Lev Kapitansky

L. V. Kapitansky "Global and unique weak solutions of nonlinear wave equations" Math. Res. Letters 1, 211–223 (1994).

M. Grillakis

M. Grillakis "Regularity and Asympt. Behavior of the Wave Eq. with a critical nonlinearity" Ann. of Math. 132 (1990), 485–509.

M. Grillakis and L. Kapitansky proved global well poseddnes in the energy class solutions and regularity of solutions for smooth inital data of the Cauchy probelm for the quintic nonlinear wave equation

$$u_{tt} - \Delta u + u^5 = 0, \quad x \in \mathbb{R}^3, t > 0$$

Initial boundary value problem under homogeneous Dirichlet's boundary conditiofor

$$u_{tt} - \Delta u + u^5 = 0, x \in \Omega \subset \mathbb{R}^3, t > 0$$

N. Burq, G. Lebeau, F. Planchon, J. AMS (2008)

Are there $p > 5$ for which solution of the problem blows up in a finite time?

Margo and Howard Levine

$$u_{tt} - \Delta u = f(u), \ v_t - \Delta v = f(v)$$

$$\Psi(t) = (\Phi(t))^{-\alpha}, \quad \alpha > 0,$$

$$\Phi(t) = \int_\Omega u(x,t)^2, \ \Phi(t) = \int_0^t \int_\Omega v(x,\eta)^2 dx d\eta + C_0$$

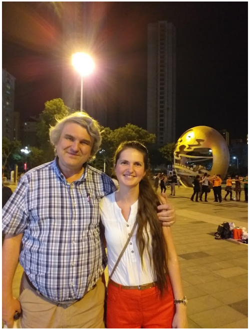
H. Levine, Arch. Rat.Mech.Anal. (1974)

Terence Tao

$$u_{tt} - \Delta u + \nabla F(u) = 0, \qquad u = (u_1, u_2, \cdots, u_m) \qquad (KG)$$

If $p > 5$ then there exists a smooth nonlinearity $V(\cdot) : \mathbb{R}^m \to \mathbb{R}$ homogeneous and positive of degree $p+1$ near infinity, such that the system of equations (KG) admits smooth compactly supported solutions that blow up in a finite time.

T. Tao, Finite-time blowup for a supercritical defocusing nonlinear wave system.Anal. PDE 9(8),1999–2030(2016)



Anna Kostienko and Sergey Zelik

$$u_{tt} - \Delta u - \alpha \Delta u_t + k|u|^p u = 0$$

V.K., S. Zelik, J. Differential Equations (2009)

Blow up of solutions and preventing blow up

$$\partial_t u + \partial_x^2 \left(\partial_x^2 u + f(u)\right) + \partial_x(u|u|^p) = g, \qquad (A)$$

where the nonlinearity f satisfies assumptions

$$|f'(v)| \le C(1+|v|^q), \quad \forall v \in \mathbb{R}, \qquad (B)$$

B.A. Bilgin, V.K., S. Zelik, J. Math. Fluid Mech. (2016)

Theorem

Let $g \in L^2(\Omega)$ and the nonlinearity f satisfy (B). Assume also that $p \geq 2q$. Then, for any sufficiently smooth solution $u(t)$ of equation (A), the following dissipative estimate holds:

$$\|u(t)\|^2_{L^2} + \int_t^{t+1} \|\partial_x^2 u(s)\|^2_{L^2} + \|u(s)\|^{p+2}_{L^{p+2}}\, ds \\ \leq C\|u_0\|^2_{L^2} e^{-\alpha t} + C(\|g\|^2_{L^2} + 1),$$

where the positive constants α and C are independent of u_0, t and g.

$p < 2q$???

Chevron pattern equations

$$\begin{cases} \tau\partial_t A = A + \Delta A - \phi^2 A - |A|^2 A - 2ic_1\phi\partial_y A + i\beta A\partial_y\phi, \\ \partial_t\phi = D_1\partial_x^2\phi + D_2\partial_y^2\phi - h\phi + \phi|A|^2 - c_2 \mathrm{Im}\left[A^*\partial_y A\right], \end{cases} \qquad (C)$$

where τ, D_1, D_2, c_1, c_2, h are non-negative constants and $\beta \in \mathbb{R}$. Here $(x, y) \in \Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary.

H. Kalantarova, V.K, O. Vantzos J.Math.Phys. (2018)

Has the initial boundary value problem for the system (C) a global solution for $c_1 > 1$?

Vladimir Mazja

V. G. Mazja, Seventy Five (Thousand) Unsolved Problems in Analysis and Partial Differential Equations, Integr. Equ. Oper. Theory (2018) 90:25

Thank You