Quantitative Gromov non-squeezing

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Hamiltonian flows in phase space $\mathbb{R}^n \times \mathbb{R}^n$

- Coordinates in $\mathbb{R}^n \times \mathbb{R}^n$: $p_1, \ldots, p_n$ (momentum) and $q_1, \ldots, q_n$ (position)
- Smooth function $H(p, q)$ gives rise to vector field

$$X_H(p,q) := -\sum_{i=1}^{n} \frac{\partial H}{\partial q_i}(p,q)\frac{\partial}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}(p,q)\frac{\partial}{\partial q_i}$$

- X_H defines a flow on $\mathbb{R}^n \times \mathbb{R}^n$
- A single trajectory in the flow $t \mapsto (p(t), q(t))$ satisfies

$$p_i'(t) = -\frac{\partial H}{\partial q_i}(p(t), q(t)), \text{ and } q_i'(t) = \frac{\partial H}{\partial p_i}(p(t), q(t))$$

- For $H = \frac{|p|^2}{2m} + V(q)$, recover Newton's equations, i.e. flow of X_H gives time evolution of position and momentum in Newtonian mechanics with potential $V(q)$.

Special properties of Hamiltonian flows in $\mathbb{R}^n \times \mathbb{R}^n$

- (Liouville's theorem) Volumes of regions in the phase space are preserved, i.e. $dp_1 \wedge dq_1 \wedge \ldots \wedge dp_n \wedge dq_n$ is preserved.
- (Hamilton, ..., Whittaker 1944) The skew-symmetric bilinear form $\omega := dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n$ is preserved - widely used in numerical analysis of molecular dynamics, celestial mechanics by way of symplectic integrators
- (Gromov non-squeezing 1985) If $A > 1$, one cannot map

$$B^{2n}(A) := \{|p|^2 + |q|^2 < A\pi^{-1}\}$$

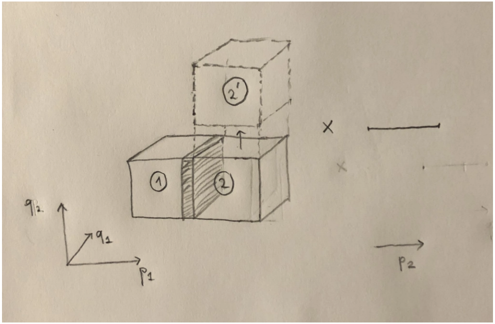
(A is the maximal area of a $2d$ cross section) into

$$Z^{2n}(1) := \{p_1^2 + q_1^2 < \pi^{-1}\}$$

by a smooth embedding preserving ω, for example time 1-maps of Hamiltonian flows (with full domain)

Hamiltonian diffeomorphisms in $\mathbb{R}^n \times \mathbb{R}^n$

- Compositions of time-1 maps of different Hamiltonian flows are called Hamiltonian diffeomorphisms: $Ham(\mathbb{R}^{2n}, \omega)$ - equivalent formulation using time dependent Hamiltonians (at least in the compactly supported case)
- (Katok 1973) For any $A > 1$, there does exist $\phi \in Ham(\mathbb{R}^{2n}, \omega)$ such that the part of $\phi(B^{2n}(A))$ that lies outside $Z^{2n}(1)$ has arbitrarily small volume
- Katok's construction (simplified version): divide $B^{2n}(A)$ into small pieces by cutting along a grid so that each small piece can be moved into $Z^{2n}(1)$ by translations, use cut-off Hamiltonian functions



This is a move that one might like to use for the transportation procedure. Try a Hamiltonian of the form $-\rho(p_1)p_2$. We get what we want for the cubes, but we also get a quite large movement in the q_1 direction in the shaded region.

Quantitative Gromov non-squeezing

- Katok embedding suggests that Gromov non-squeezing might be difficult to detect, for example in computer simulations?
- (Guth) Can we bound the volume sticking out if we put a bound on the Lipschitz constant?
- (Sackel-Song-V.-Zhu) Yes! For $n = 2$, if the Lipschitz constant is L, then $\frac{c(A)}{L^2}$ volume needs to stick out, where $c(A)$ is asymptotic to $const.A^2$ as $A \to \infty$.
- Optimal? Currently we have no construction that comes close.
- This result easily follows from our obstructive result for the Minkowski dimension question, where in some range of A we can also prove optimality. I will focus on that question.

Interlude: Symplectic manifolds

- The following equation (which is true) characterizes X_H fully:

  $$\omega(\cdot, X_H) = dH.$$

  Note that the RHS is coordinate independent, so we don't need coordinates to turn H into X_H, we only need ω.
- If we had a space M constructed by gluing open subsets of $\mathbb{R}^{2n}$ where the gluing maps preserve ω, then we would be able to consider Hamiltonian flows given by functions on M.
- Denoting the glued 2-form on M by Ω, we have $\Omega^n \neq 0$ and $d\Omega = 0$.
- (Darboux theorem) Conversely, these two properties imply that every point in M admits coordinate charts where Ω looks like ω $\rightarrow$ modern definition of symplectic form

Symplectic forms arise naturally in different contexts

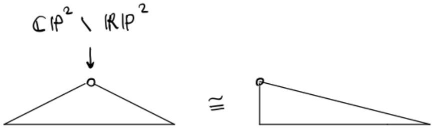
- T^*X, X smooth manifold
- $\mathbb{C}^n$, $\mathbb{C}P^n$ and their smooth complex submanifolds
- Symplectic reduction - possible to start with a simple space like $\mathbb{R}^{2n}$ and end up with a globally interesting space by taking quotients by Hamiltonian actions of Lie groups
- Coadjoint orbits of Lie groups
- Some moduli spaces ...
- It seems to be the case that for finding symplectomorphism trying to do things by hand (moving boxes, pushing things in desired directions) does not capture what is really possible.

Minkowski dimension question

- Main question: what is the smallest Minkowski dimension of a closed $E \subset B^{2n}(A)$ such that $B^{2n}(A) \setminus E$ symplectically embeds into $Z^{2n}(1)$?
- Here Minkowski dimension stands for the lower Minkowski dimension of $E \subset \mathbb{R}^{2n}$ - defined for any subset of $B^{2n}(A)$.
- Heuristically, $E \subset \mathbb{R}^N$ having Minkowski dimension $d \in \mathbb{R}$ means that as $\epsilon \to 0$, the volume of the ϵ-neighborhood of E behaves as $c\epsilon^{N-d}$, for some constant $c > 0$.
- If S is a submanifold, then we recover the usual dimension. The Minkowski dimension of the Cantor set is $log(2)/log(3)$; of $\{0\} \cup \{1, 1/2, 1/3, \ldots\}$ is $1/2$.
- Let $n = 2$ and drop superscript $2n$'s from notation from now on. Our results currently do not extend to higher dimensions. From the obstructive side the issue is lack positivity of intersection for J-holomorphic curves.

Constructive side I

- $\mathbb{CP}^2$ with Fubini-Study form, $\mathbb{RP}^2$ the real part and $\mathbb{CP}^1 := \{z_3 = 0\} \subset \mathbb{CP}^2$ has area 2
- (Oakley-Usher) $\mathbb{CP}^2 \setminus \mathbb{RP}^2$ admits a Hamiltonian torus action with moment map image as shown



such that the preimage of the slope $1/2$ edges is $\mathbb{CP}^1 \setminus \mathbb{RP}^2$
- Using Karshon-Lerman's extension of Delzant theorem to open symplectic toric manifolds:

$$B(2) \setminus L \simeq \mathbb{CP}^2 \setminus (\mathbb{CP}^1 \cup \mathbb{RP}^2) \simeq E(4,1) \setminus Z,$$

where $L \subset \mathbb{R}^4$ is a Lagrangian subspace, $E(4,1)$ is an ellipsoid and $Z = \{p_2 = q_2 = 0\}$.

Constructive side II

- Theorem (SSVZ): $B(2) \setminus L$ embeds into $Z(1)$.
- Explicit formula for the moment map in Remark 3.2 of OU.
- Biran-Giroux decomposition: $\mathbb{CP}^2 \simeq D^*\mathbb{RP}^2/\text{bdry red}$.
- Consider the spherical pendulum system with zero gravity: T^*S^2 and (energy, angular momentum around a fixed direction) gives an integrable system. Then take $\mathbb{Z}/2$ quotient.
- In the paper we find an explicit symplectomorphism using an observation of Opshtein.
- The discovery was made using an entirely different story during conversations with Mikhalkin (next two slides).
- The embedding of $B(2) \setminus N_\epsilon(L)$ does not extend to a symplectic embedding of $B(2)$ into $\mathbb{R}^4$ for sufficiently small (but not that small) ϵ.

Toric degeneration of $\mathbb{CP}^2$ to $\mathbb{CP}^2(1,1,4)$ I

- The weighted projective space $\mathbb{CP}^3(1,1,1,2)$ has a single orbifold point and in its complement there is a natural symplectic form Ω.
- Consider the pencil:

$$\Xi_{[t:s]} := \{tz_1z_2 - (t-s)z_3^2 - sz_4 = 0\}$$

- Doing a Nash blow-up and removing a fiber we obtain

$$w := \frac{s}{t} : P - \Xi_{[1:0]} \to \mathbb{C}.$$

- w has no critical points if we exclude the orbifold point.
- We have $w^{-1}(1) \simeq \mathbb{CP}^2$ and $w^{-1}(0) \simeq \mathbb{CP}^2(1,1,4)$
- Moreover, these identifications can be made symplectic where we use standard symplectic structures on the RHS.

Toric degeneration of $\mathbb{CP}^2$ to $\mathbb{CP}^2(1,1,4)$ II

- Ω gives rise to an Ehressmann connection for w restricted to non-orbifold points.
- Therefore, we obtain a parallel transport symplectomorphism

$$w^{-1}(1) \setminus \text{ (whatever converges to the orbifold point)} \simeq \\ w^{-1}(0) \setminus \text{ (the orbifold point)}$$

- The singularity of w at the orbifold point is the simplest Wahl singularity and its vanishing cycle (i.e. stuff that converges to the orbifold point) is known to be a real projective plane.
- One can also trace the image of a $\mathbb{CP}^1(1,4) \subset \mathbb{CP}^2(1,1,4)$ and more or less see that it's the one half a complex line in $\mathbb{CP}^2$ which intersects our $\mathbb{RP}^2$ along an $\mathbb{RP}^1$.
- This suggests the result we proved above

Obstructive side I

- Theorem (SSVZ): For $A > 1$, the Minkowski dimension of a closed subset E such that $B(A) \setminus E$ symplectically embeds into $Z(1)$ is at least 2.
- The result is optimal for $2 \geq A > 1$ as our construction above shows.
- The proof has two main ingredients: the argument in the proof of Gromov non-squeezing and Gromov's waist inequality. These are very substantial ingredients.
- We also need an elementary bound on the volume of small tubular neighborhoods of minimal surfaces (Heintze-Karcher inequality).

Obstructive side II

- Take such an embedding Φ. We need to show that the volume of the $\delta << 1$ neighborhood of E behaves like $const.\delta^2$.
- Fix $\delta << 1$. For any $\epsilon < \delta$, $a > 1$ and $\alpha > 0$, following the argument in the proof of Gromov non-squeezing, we find a continuous function

$$f : B(A-\alpha) \to \mathbb{R}^2$$

with the following properties

1. Outside of $\overline{N_\epsilon(E)}$, f is smooth with no critical points.
2. For all $y \in \mathbb{R}^2$,

$$(B(A-\alpha) \setminus \overline{N_\epsilon(E)}) \cap f^{-1}(y)$$

is a complex submanifold of area less than a.

- Let $\alpha' := \alpha + 2(\delta - \epsilon)$

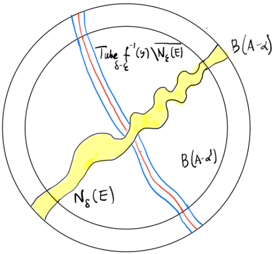
Obstructive side III

- Gromov's waist inequality will give us a special $y \in \mathbb{R}^2$ such that the volume of the $\delta - \epsilon$ neighborhood of $f^{-1}(y)$ in $B(A - \alpha')$ is at least $\pi(A - \alpha')(\delta - \epsilon)^2 + o((\delta - \epsilon)^2)$.
- The HK inequality and the area bound says that the volume of the $\delta - \epsilon$ tubular neighborhood of $(B(A - \alpha) \setminus \overline{N_\epsilon(E)}) \cap f^{-1}(y)$ in $\mathbb{R}^4$ is at most $\pi a(\delta - \epsilon)^2$.
- This tubular neighborhood and the δ-neighborhood of E in $B(A - \alpha)$ cover the $\delta - \epsilon$ neighborhood of $f^{-1}(y)$ in $B(A - \alpha')$ (next slide).
- Hence we get a lower bound

$$\pi(A - \alpha')(\delta - \epsilon)^2 - \pi a(\delta - \epsilon)^2 + o((\delta - \epsilon)^2)$$

  on the volume of the δ neighborhood of E (in $B(A - \alpha)$ and therefore also in $\mathbb{R}^4$)
- Now, we let ϵ to 0, a to 1 and α to 0 to get the final result.

Obstructive side IV



- Let $p \in B(A-\alpha')$ and $dist(p, f^{-1}(y)) \leq \delta - \epsilon$. If $dist(p, E) \leq \delta$, good. Otherwise, let $z \in f^{-1}(y)$ be a closest point, which has to exist.
- By triangle ineq. we have $dist(z, E) > \epsilon$ and therefore near z $f^{-1}(y)$ is a submanifold.
- We get that the straight line from z to p is perpendicular to $f^{-1}(y)$ and therefore p is in the desired tubular neighborhood

Gromov's waist inequality

- Theorem (Gromov): Let $f : S^n \to \mathbb{R}^k$ be a continuous map where $n \geq k$. Here we are thinking of S^n as the unit sphere in $\mathbb{R}^{n+1}$. An example of such a map is pr_k, which projects to the first k-coordinates. Then, there exists a $y \in \mathbb{R}^k$ such that

$$vol(N_t(f^{-1}(y)) \geq vol(N_t(pr_k^{-1}(0)),$$

for every (!) $t \geq 0$. (weak Borsuk-Ulam for $n = k$, $t = \pi/2$)
- We need a similar result for the ball in our proof. This result is deduced by Akopyan-Karasev using the Archimedes map (pr_n with target replaced with its image):

$$S^{n+1} \to B^n,$$

which is measure preserving and contracting.
- The inequality one gets is not optimal as under the Archimedes map the image of the t-neighbhorhood does not cover the t-neighborhood of the image. As t tends to 0, we approach to optimality.

Questions

- Is the obstructive result optimal for $A > 2$?
- Is our bound on the Minkowski content optimal?
- What happens if we require the embedding to "extend" to the ball in the Minkowski dimension question?
- The Gromov capacity of the ball is halved if we remove a Lagrangian subspace. Due to results of Traynor, it stays the same if we remove a complex subspace. What is the symplecticity to capacity function?
- Higher dimensions??
- Lipschitz question???
- Thank you for listening!