Nonlinear Algebra in Particle Physics

Based on:

- *Vector Spaces of Generalised Euler Integrals*
  with D. Agostini, A.-L. Sattelberger, and S. Telen, ArXiv:2208.089
- *Principal Landau Determinants*
  with S. Mizera and S.Telen, ArXiv:2311.16219

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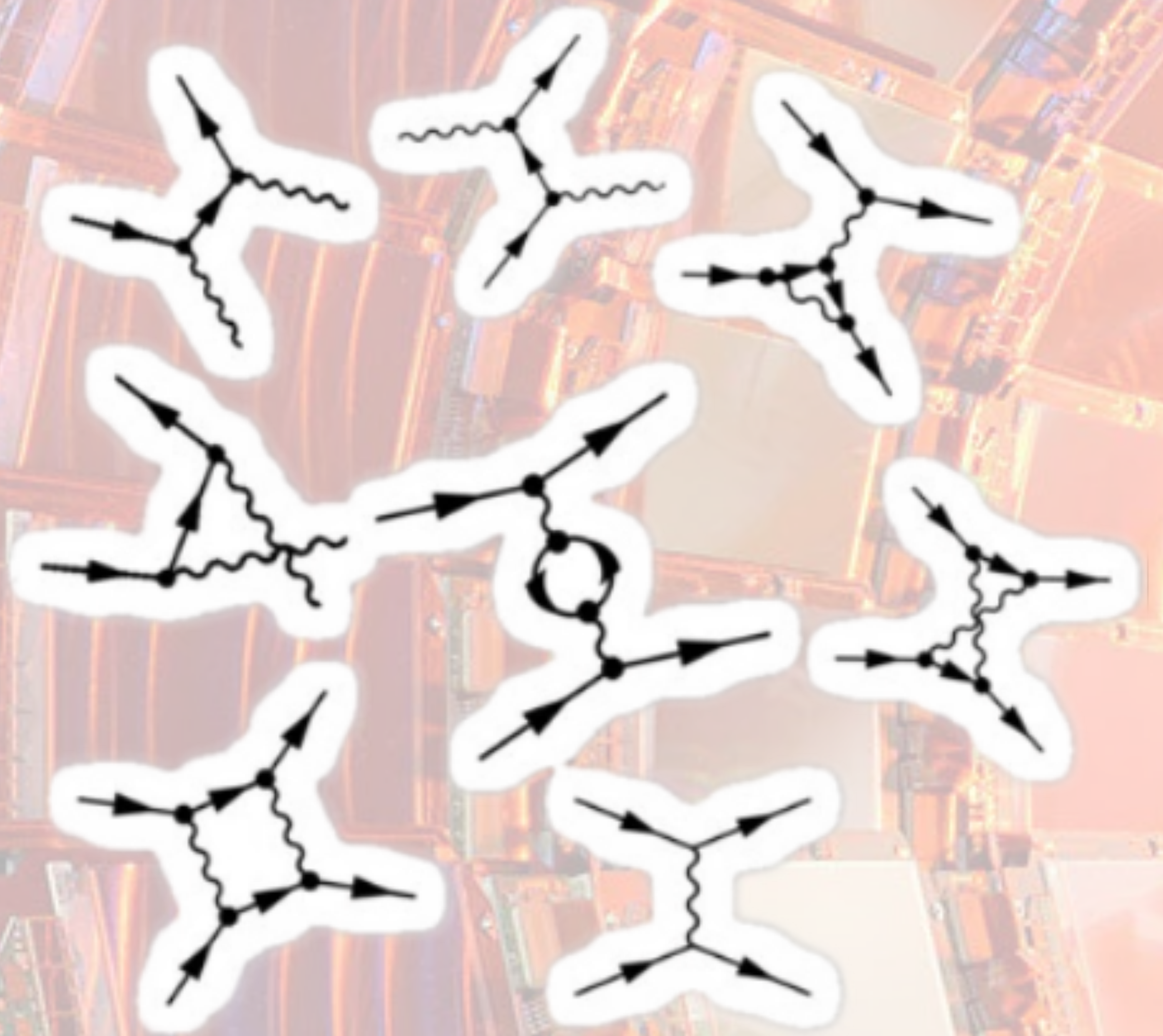
Koç University Math Seminar

December 14, 2023

Outlook

1. Scattering amplitudes in a (small) nutshell
2. Feynman integrals as generalised Euler integrals
3. Singularities of Feynman integrals

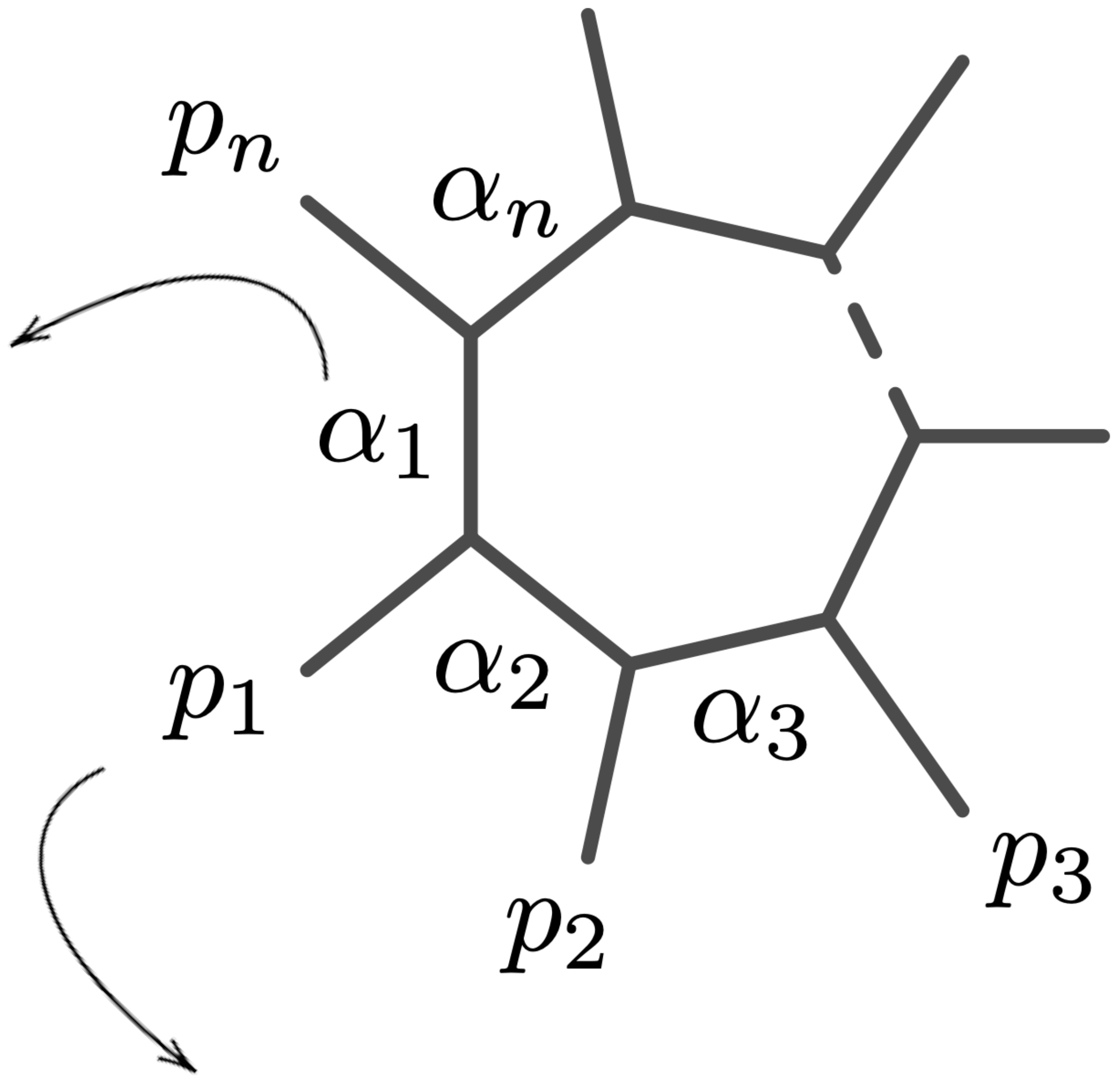
Particle Physics: Scattering Amplitudes



$$A = \sum_{G \in \mathscr{G}} I_G$$

Feynman Integrals

$G = (V, E)$ connected undirected graph



$\alpha_i \in \mathbb{C}^*$

$p_i \in \mathbb{R}^{1,D-1}$

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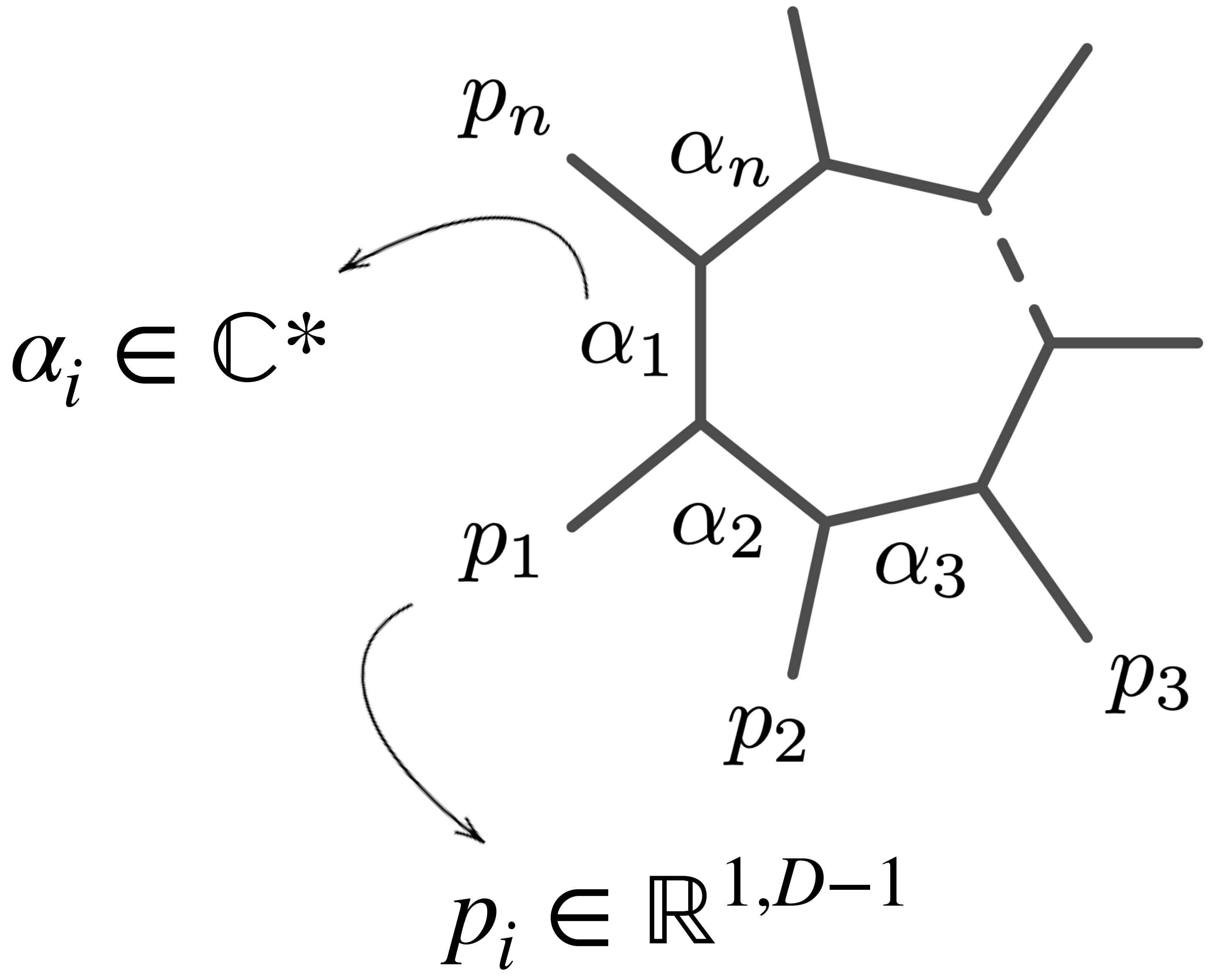

$$I_{b_1,\ldots,b_n} = \# \int_0^\infty \frac{\alpha_1^{b_1} \cdots \alpha_n^{b_n}}{(\underbrace{\mathcal{U}_G + \mathcal{F}_G}_{\mathcal{G}_G})^{D/2}} \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{\mathrm{d}\alpha_n}{\alpha_n}$$

Graph polynomial

[Lee-Pomeransky, '13]

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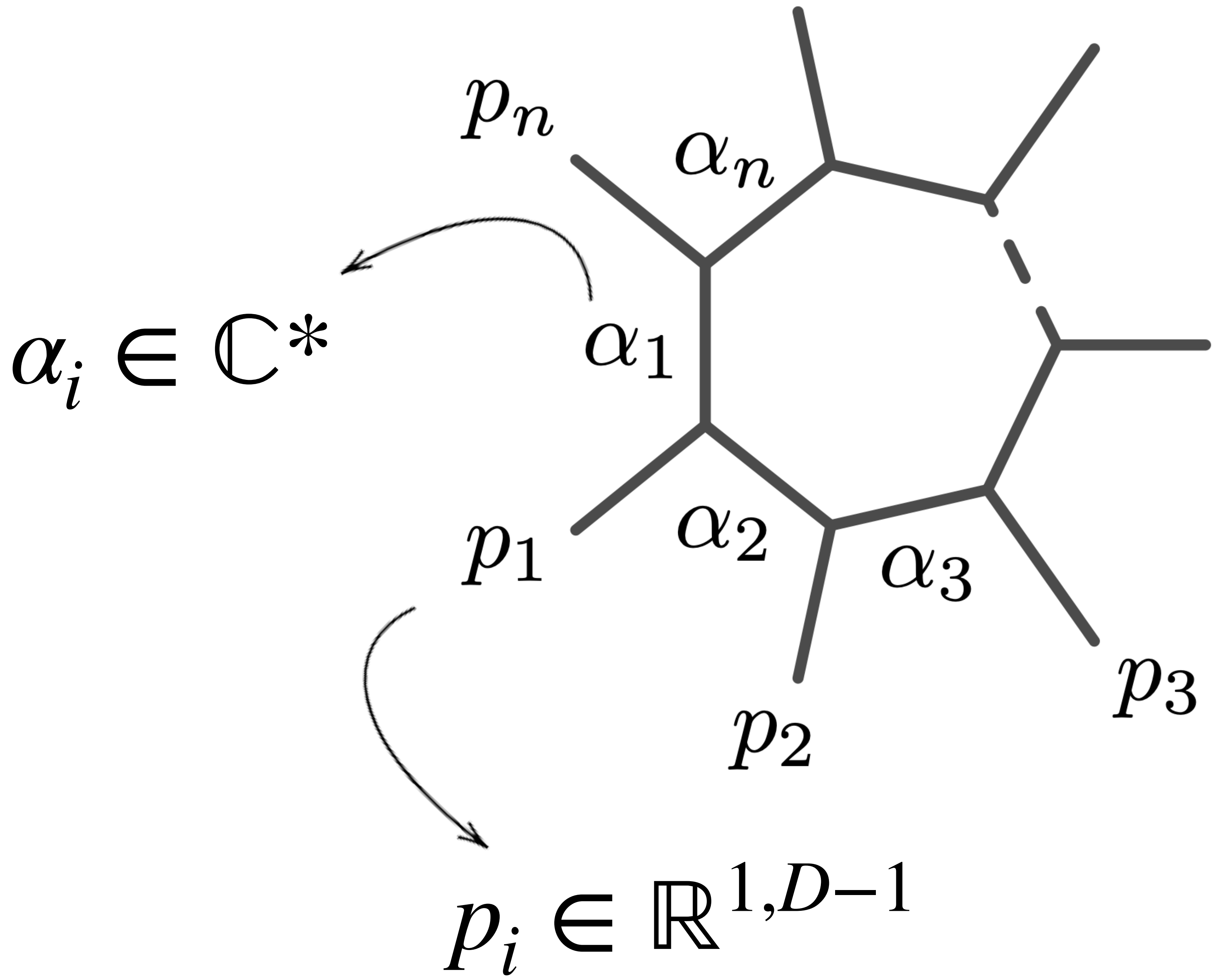

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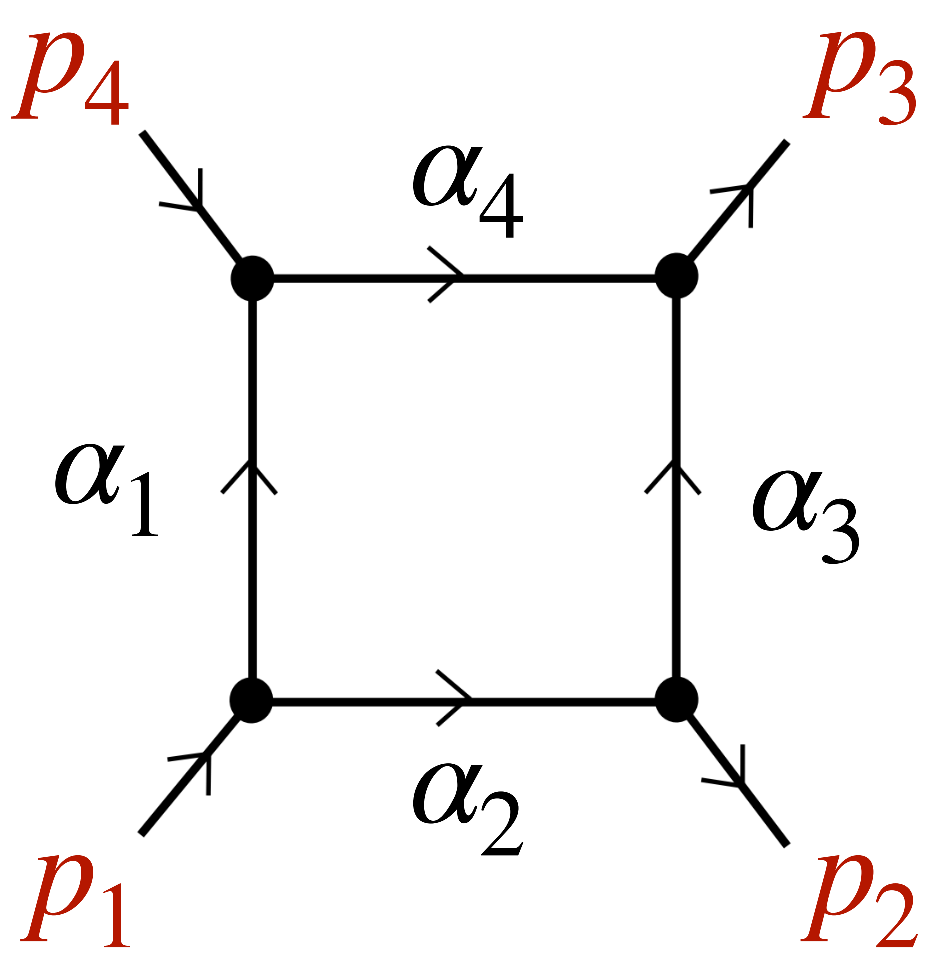
Graph polynomial

[Lee-Pomeransky, '13]

$\mathcal{U}_G, \mathcal{F}_G$ homogeneous polynomials in the variables α with coefficients in the kinematic space $\mathcal{K} \subset \mathbb{C}^m$

Feynman Integrals: box diagram



$$n = 4,\ E = 4,\ L = 1$$

$$\alpha_e \in \mathbb{C}^*, m_e \in \mathbb{R}_{\geq 0}, \quad e = 1,\ldots,E$$

$$p_i \in \mathbb{R}^{1,D-1}, b_i \in \mathbb{N}$$

$$\mathcal{U} \ = \ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$\begin{aligned}\mathcal{F} \ = \ & p_1^2\alpha_1\alpha_2 + p_2^2\alpha_2\alpha_3 + p_3^2\alpha_3\alpha_4 + p_4^2\alpha_1\alpha_4 + (p_1+p_2)^2\alpha_1\alpha_3 + (p_2+p_3)^2\alpha_2\alpha_4 + \\ & -(m_1^2\alpha_1 + m_2^2\alpha_2 + m_3^2\alpha_3 + m_4^2\alpha_4)\mathcal{U}\end{aligned}$$

$$I_{b_1,b_2,b_3,b_4} = \# \int_0^\infty \frac{\alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3}\alpha_4^{b_4}}{\mathcal{G}^{D/2}} \frac{\mathrm{d}\alpha}{\alpha}$$

Vector Spaces of Generalised Euler Integrals

Generalised Euler Integrals [GKZ]

$$\int_\Gamma f^s \alpha^\nu \frac{\mathrm{d}\alpha}{\alpha} = \int_\Gamma \left(\prod_{j=1}^{\ell} f_j^{s_j}\right) \cdot \left(\prod_{i=1}^{n} \alpha_i^{\nu_i}\right) \frac{\mathrm{d}\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{\mathrm{d}\alpha_n}{\alpha_n}$$

- $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^*)^n$
- $f = (f_1, \ldots, f_\ell) \in \mathbb{C}[\alpha, \alpha^{-1}]^\ell$
- $s = (s_1, \ldots, s_\ell) \in \mathbb{C}^\ell, \quad \nu = (\nu_1, \ldots, \nu_n) \in \mathbb{C}^n$
- $\Gamma \in H_n(X, \omega)$, where $\omega = \mathrm{d}log(f^s \alpha^\nu)$

→ Twisted de Rham homology group

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$$X := \{\, \alpha \in (\mathbb{C}^*)^n \mid f_1(\alpha)\cdots f_\ell(\alpha) \neq 0 \,\} = (\mathbb{C}^*)^n \setminus V(f_1 \cdots f_\ell)$$

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Twisted de Rham homology group

Feynman integrals: $\ell = 1, f =$ Graph polynomial

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Vector spaces of Generalised Euler Integrals

$$V_\Gamma := \mathrm{Span}_{\mathbb{C}}\left\{ [\,\Gamma\,] \longmapsto \int_\Gamma f^{s+a}\,\alpha^{\nu+b}\,\frac{\mathrm{d}\alpha}{\alpha} \right\}_{(a,b)\in\mathbb{Z}^\ell\times\mathbb{Z}^n}$$

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Mastrolia, Mizera

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→ GKZ systems

Matsubara-Heo, Chestnov, …

Theorem (Agostini, F., Sattelberger, Telen):

Let $f = (f_1, \ldots, f_\ell) \in \mathbb{C}[\alpha, \alpha^{-1}]^\ell$ be Laurent polynomials with fixed monomial supports and generic coefficients. Consider V_Γ, V_{c^*} with generic choices of parameters each. Then

$$\dim_{\mathbb{C}}(V_\Gamma) \;=\; \dim_{\mathbb{C}}(V_{z^*}) \;=\; (-1)^n \cdot \chi(X)\,.$$

Topological Euler characteristic

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Topological Euler characteristic

Computing Euler characteristics

Theorem (Huh): $|\chi(X)|$ equals the number of critical points of

$$L \;=\; \log(f^s \alpha^\nu) \;=\; \sum_{j=1}^{\ell} s_j \log f_j \,+\, \sum_{i=1}^{n} \nu_i \log \alpha_i$$

for general s, ν .

Solving rational function equations using Homotopy Continuation.jl

```julia
using HomotopyContinuation

@var α[1:3], s, m[1:3], u[1:4]
f = (1 - m[1]*α[1] - m[2]*α[2] - m[3]*α[3])*
    (α[1]*α[2] + α[2]*α[3] + α[3]*α[1]) + s*α[1]*α[2]*α[3]

W = u[1] * log(f) + dot(u[2:4], log.(α))
dW = System(differentiate(W, α), parameters = [s; m; u])

Crit = monodromy_solve(dW)
crt = certify(dW, Crit)
println(ndistinct_certified(crt))
```

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GKZ systems $(\ell = 1)$

$$A = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ m_1 & m_2 & \cdots & m_s \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} \in \mathbb{Z}^{n\times s}, \quad \text{rank}(A) = n$$

$$\alpha^{m_i} = \alpha_1^{m_{1i}} \cdots \alpha_n^{m_{ni}}$$

$$f_A(\alpha; z) = z_1\,\alpha^{m_1} + z_2\,\alpha^{m_2} + \cdots + z_s\,\alpha^{m_s}$$

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

$$z_i \in \mathbb{C}$$

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$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

$$z_i \in \mathbb{C}$$

$$\boxed{X_{A,z} = (\mathbb{C}^*)^n \setminus V_{(\mathbb{C}^*)^n}(f_A(\alpha; z)) = \{\alpha \in (\mathbb{C}^*)^n \,:\, f_A(\alpha; z) \neq 0\}}$$

GKZ systems

Consider the Weyl algebra $D_A = \mathbb{C}[z_\alpha \mid \alpha \in A]\langle \partial_{m_i} \mid m_i \in A \rangle$

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Definition

$$I_A := \langle \partial^u - \partial^v \,|\, u - v \in \ker(A),\, u, v \in \mathbb{N}^A \rangle \triangleleft \mathbb{C}[\partial_{m_i} \,|\, m_i \in A]$$

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Let $\kappa = (-\nu, s)^\top \in \mathbb{C}^{n+\ell}$

$$J_{A,\kappa} := \langle\, (A\theta - \kappa)_i,\; i = 1,\ldots,n+\ell \,\rangle,$$

where $\theta := (\theta_{m_i})_{m_i \in A}$ and $\theta_{m_i} = c_{m_i}\partial_{m_i}$

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$$H_A(k) \;:=\; I_A + J_{A,\kappa} \;\triangleleft\; D_A$$

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Example!!!

Theorem (Cauchy, Kovalevskaya, Kashiwara):

The dimension of the space of solutions of a D-ideal I on a simply connected domain U outside the singular locus $\mathrm{Sing}(I)$ is equal to the holonomic rank of I.

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Let $z^* \in \mathbb{C}^A$ be such that and let κ be generic. For any simply connected domain $U_{z^*} \ni z^*$ outside the singular locus, we have that

$$\dim_{\mathbb{C}}(V_{z^*}) = \dim_{\mathbb{C}(z)}(R_A/(R_A \cdot H_A(\kappa))) = |\chi(X_{A,z^*})| = \mathrm{vol}(\mathrm{Newt}(f_A(\alpha, z^*)))$$

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For GKZ systems, we have

$$\text{Sing}(H_A(\kappa)) = \{E_A(z) = 0\}$$

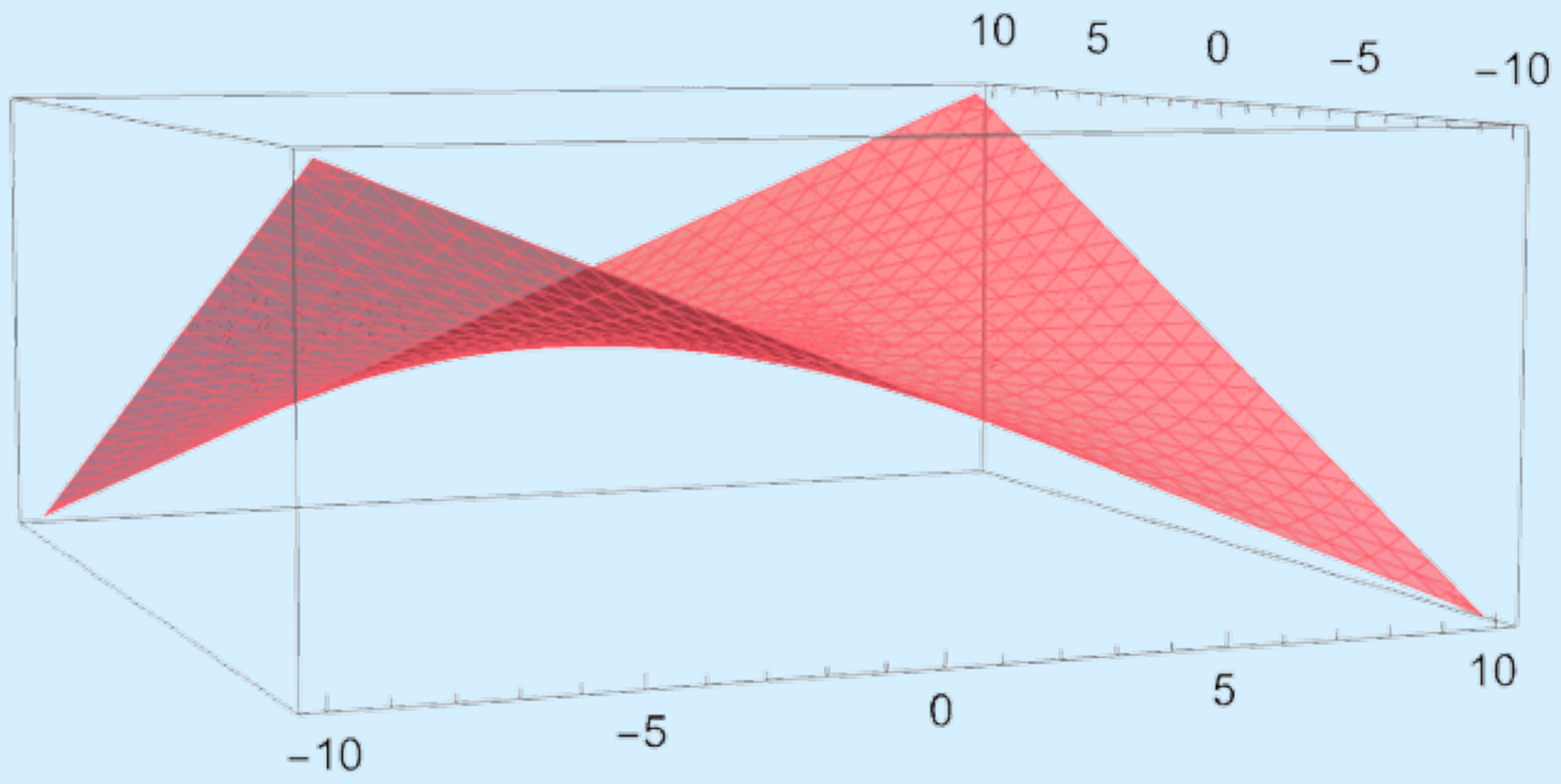
Principal A-determinant

Principal Landau Determinants

A-discriminants

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$



$$f_A(\alpha, z) \; = \; z_1 + z_2\,\alpha_1 + z_3\,\alpha_2 + z_4\,\alpha_1\alpha_2$$

$$\Delta_A \; = \; \det\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = z_1 z_4 - z_2 z_3$$

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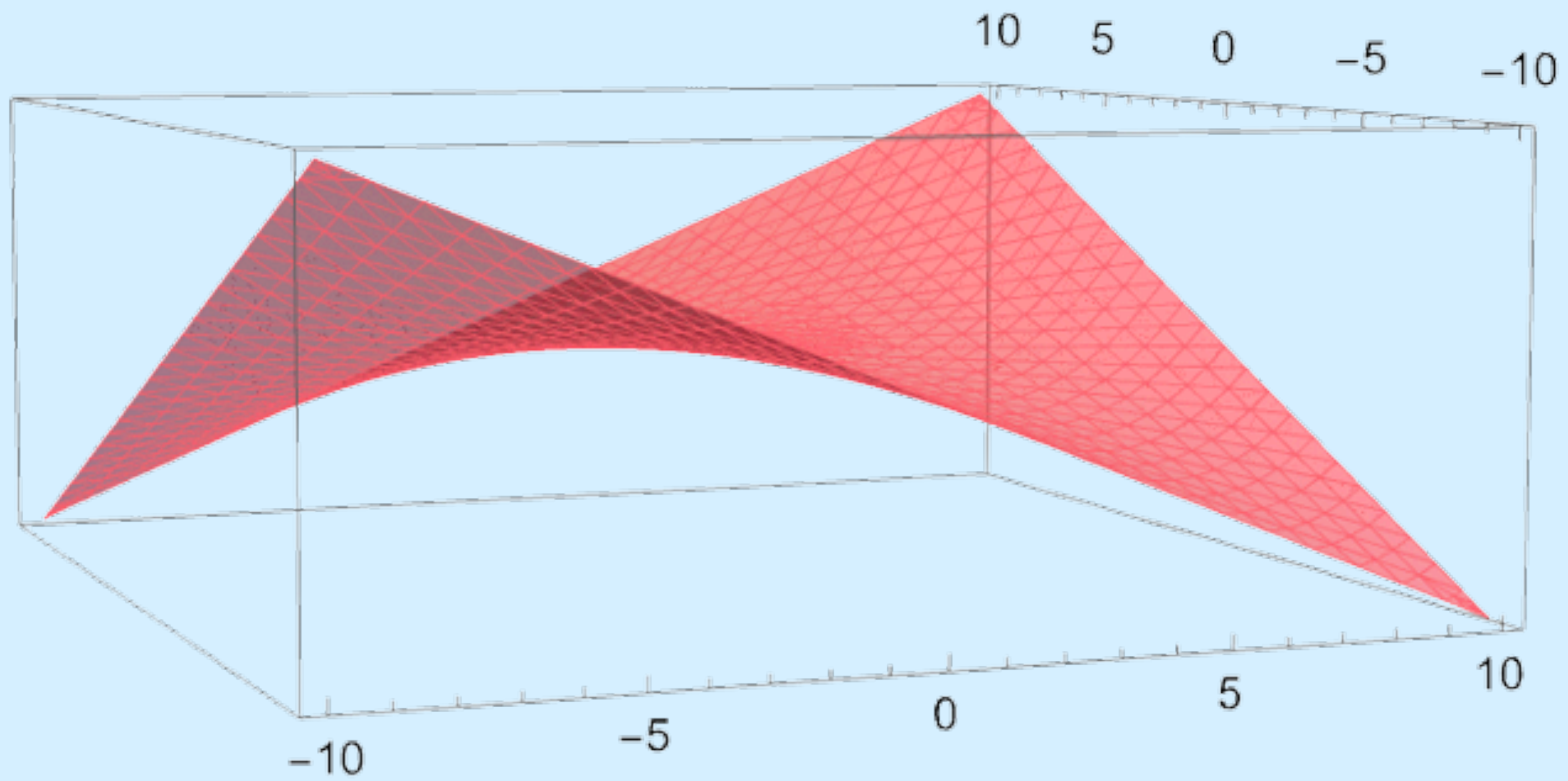
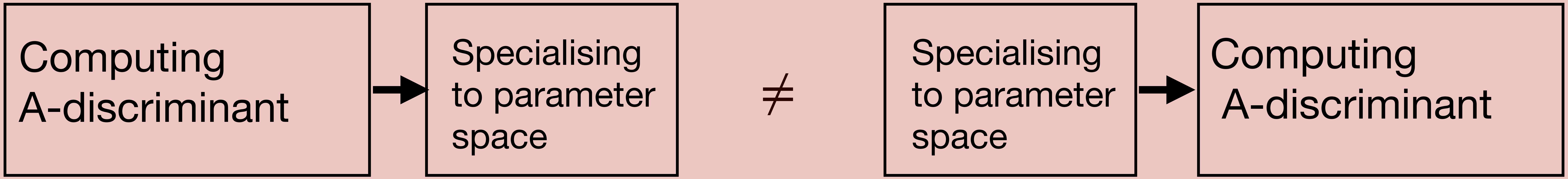


$$\nabla_A^\circ = \left\{ z \in \mathbb{C}^s \,:\, \exists \alpha \in (\mathbb{C}^*)^n \text{ s.t. } f_A(\alpha; z) = \partial_\alpha f_A(\alpha; z) = 0 \right\}$$

$$\partial_\alpha = (\partial_{\alpha_1}, \ldots, \partial_{\alpha_n})$$

Definition: The A-discriminant variety $\nabla_A = \overline{\nabla_A^\circ}$ records values of z for which $V_{A,z}$ is a singular hypersurface.

Remark


Computing A-discriminant
Specialising to parameter space
≠
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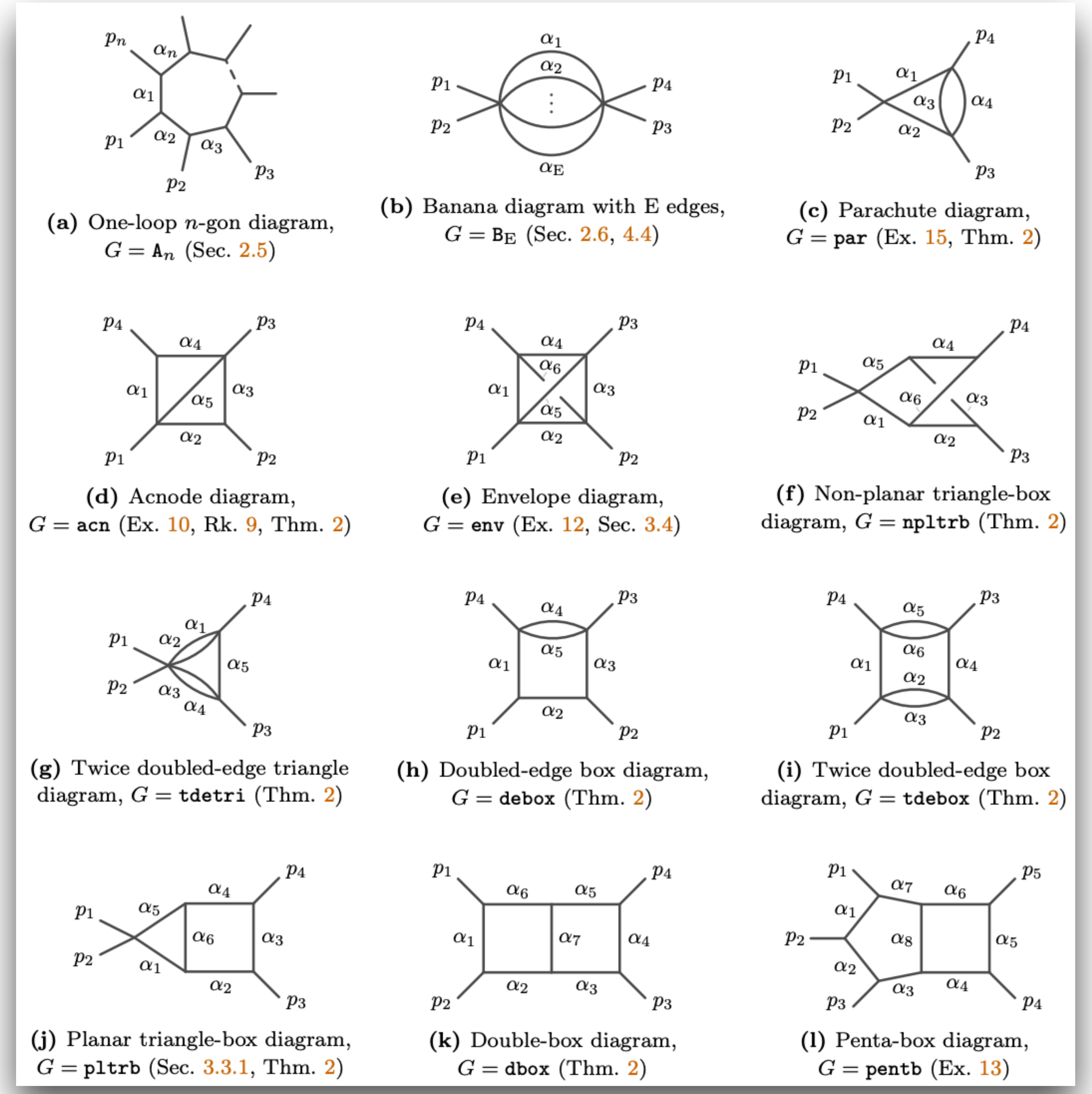
Remark

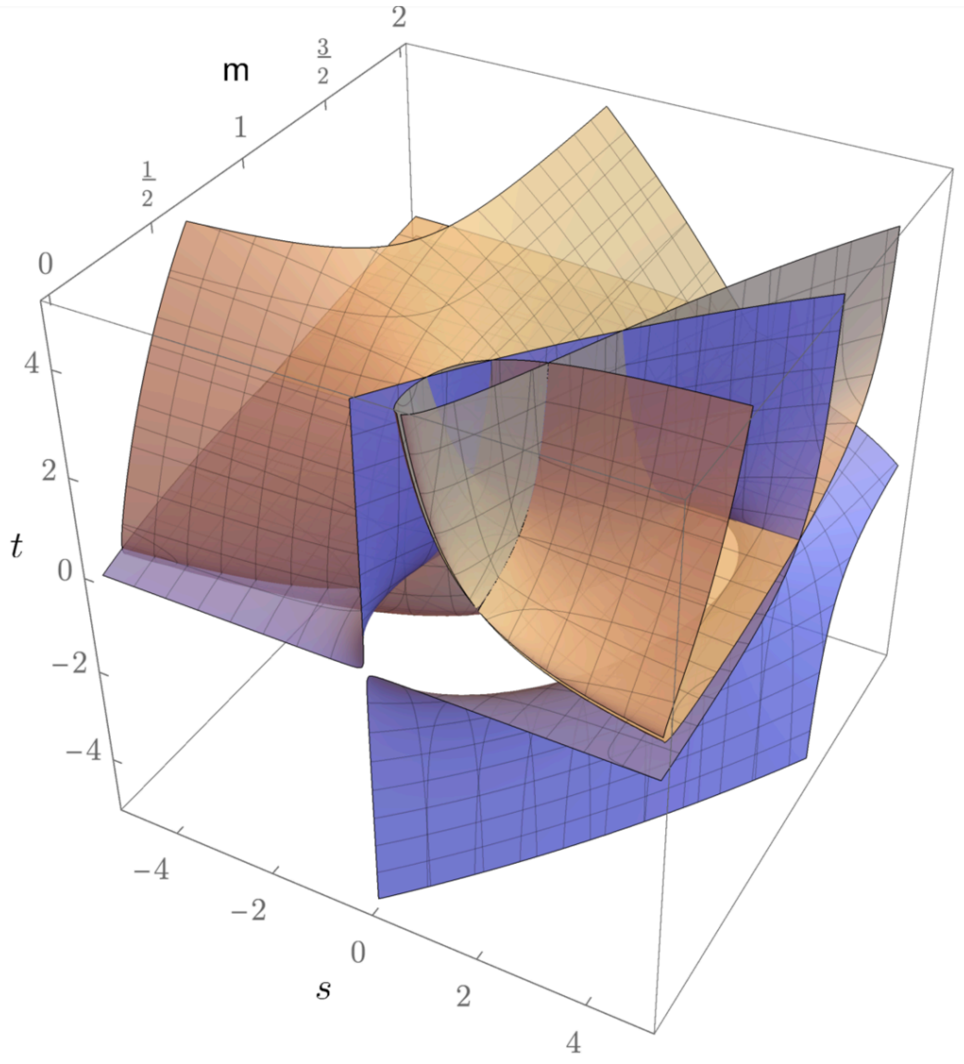
Computing A-discriminant → Specialising to parameter space $\neq$ Specialising to parameter space → Computing A-discriminant



Landau discriminants

Sebastian Mizera & Simon Telen

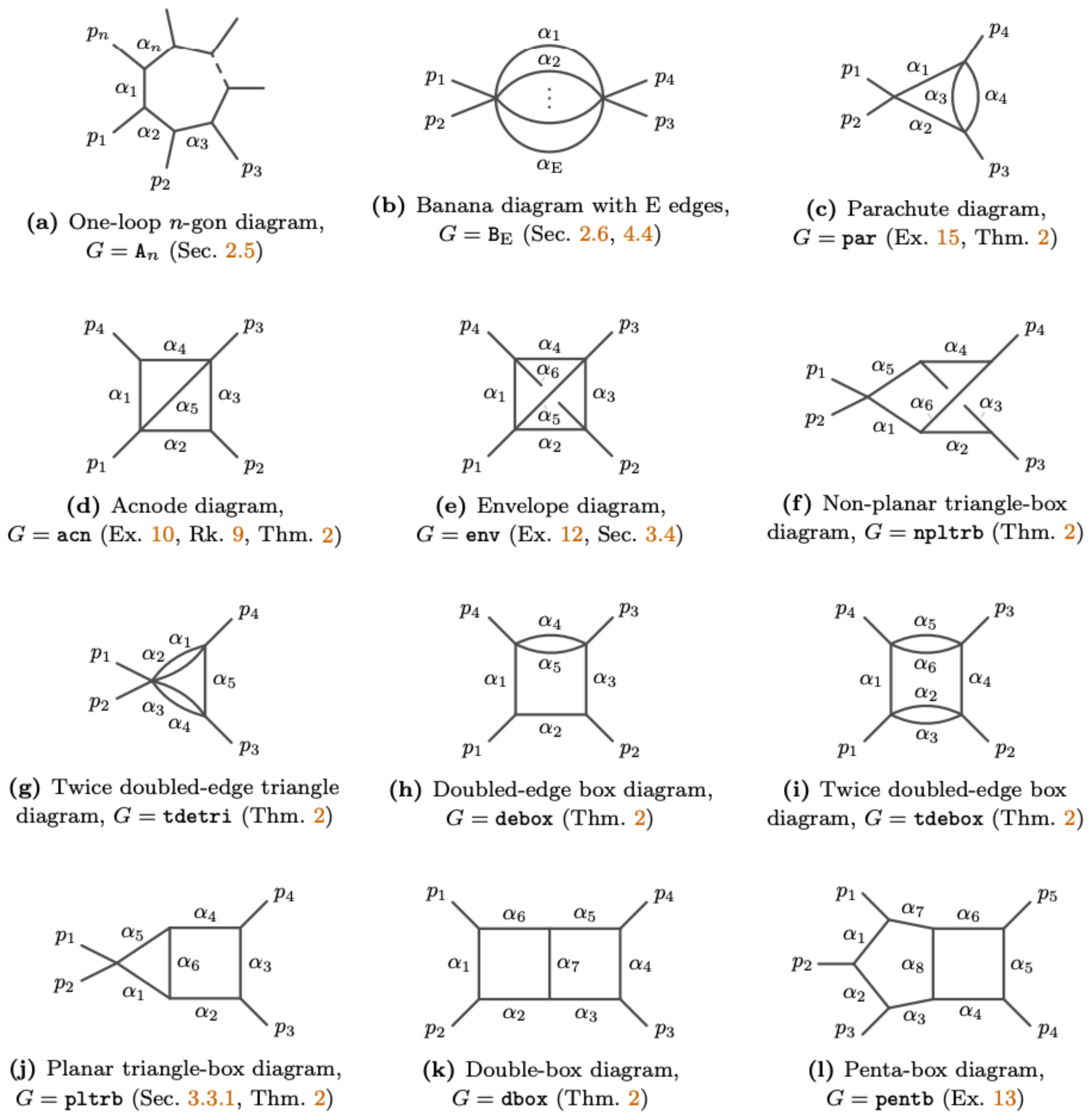
Journal of High Energy Physics **2022**, Article number: 200 (2022)



Landau.jl

julia

χ VS volume



(a) One-loop n-gon diagram, $G = \mathtt{A}_n$ (Sec. 2.5)

(b) Banana diagram with E edges, $G = \mathtt{B}_{\mathrm{E}}$ (Sec. 2.6, 4.4)

(c) Parachute diagram, $G = \mathtt{par}$ (Ex. 15, Thm. 2)

(d) Acnode diagram, $G = \mathtt{acn}$ (Ex. 10, Rk. 9, Thm. 2)

(e) Envelope diagram, $G = \mathtt{env}$ (Ex. 12, Sec. 3.4)

(f) Non-planar triangle-box diagram, $G = \mathtt{npltrb}$ (Thm. 2)

(g) Twice doubled-edge triangle diagram, $G = \mathtt{tdetri}$ (Thm. 2)

(h) Doubled-edge box diagram, $G = \mathtt{debox}$ (Thm. 2)

(i) Twice doubled-edge box diagram, $G = \mathtt{tdebox}$ (Thm. 2)

(j) Planar triangle-box diagram, $G = \mathtt{pltrb}$ (Sec. 3.3.1, Thm. 2)

(k) Double-box diagram, $G = \mathtt{dbox}$ (Thm. 2)

(l) Penta-box diagram, $G = \mathtt{pentb}$ (Ex. 13)

$(|\chi(V_A(\mathscr{E}))|, \mathrm{vol}(A(\mathscr{E})))$

G	$\mathcal{K}$	$\mathcal{E}^{(\mathrm{M}_i,0)}$	$\mathcal{E}^{(0,\mathrm{m}_e)}$	$\mathcal{E}^{(0,0)}$
$\mathtt{A}_4$	(15, 15)	(11, 11)	(11, 15)	(3, 3)
$\mathtt{B}_4$	(15, 35)	(1,1)	(15,35)	(1,1)
par	(19, 35)	(4, 8)	(13, 35)	(1, 3)
acn	(55, 136)	(20, 54)	(36, 136)	(3, 9)
env	(273, 1496)	(56, 262)	(181, 1496)	(10, 80)
npltrb	(116, 512)	(28, 252)	(77, 512)	(5, 61)
tdetri	(51, 201)	(4, 18)	(33, 201)	(1, 5)
debox	(43, 96)	(11, 33)	(31, 96)	(3, 10)
tdebox	(123, 705)	(11, 113)	(87, 705)	(3, 41)
pltrb	(81, 417)	(16, 201)	(61, 417)	(4, 80)
dbox	(227, 1422)	(75, 903)	(159, 1422)	(12, 238)
pentb	(543, 4279)	(228, 3148)	(430, 4279)	(62, 1186)

Principal A-determinant [GKZ]

$$P := \operatorname{conv}(A) \subset \mathbb{R}^n$$

$$E_A = \prod_{Q \in F(A)} \Delta_{A \cap Q}^{e_Q}$$

$e_\Gamma \in \mathbb{N}$

$$A \cap Q = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ m_1 & m_2 & \cdots & m_s \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

$m_i \in Q$

$F(A)$: Set of faces of P

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$F(A)$: Set of faces of P

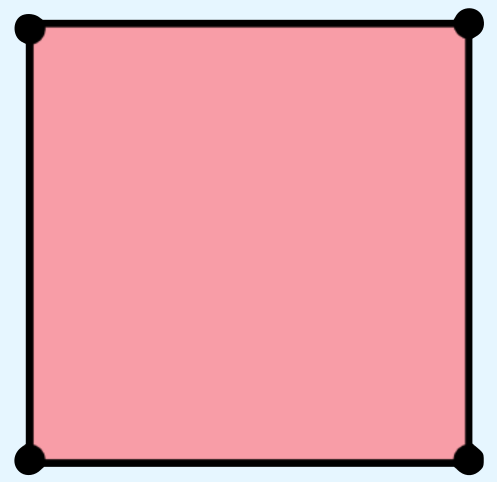
Theorem (Amendola, Bliss, Burke, Gibbons, Helmer, Hoşten, Nash, Rodriguez, Smolkin, 2012):

$$|\chi(V_{A,z^*})| = \mathrm{vol}(A) \quad \Longleftrightarrow \quad z^* \in \mathbb{C}^s \setminus \{E_A(z) = 0\}$$

Moreover, when $E_A(z) = 0$, we have $|\chi(V_{A,z})| < \mathrm{vol}(A)$.

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

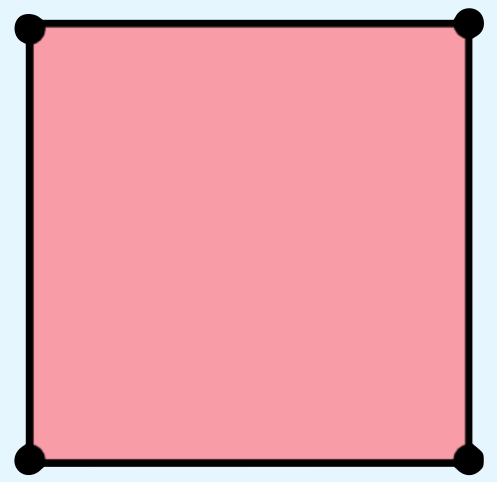


$$f_A(\alpha, z) \;=\; z_1 + z_2\,\alpha_1 + z_3\,\alpha_2 + z_4\,\alpha_1\alpha_2$$

$$E_A \;=\; z_1 \cdot z_2 \cdot z_3 \cdot z_4 \cdot (z_1 z_4 - z_2 z_3)$$

Example

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$



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Computing principal A-determinant → Specialising to parameter space $\neq$ Specialising to parameter space → Computing principal A-determinant

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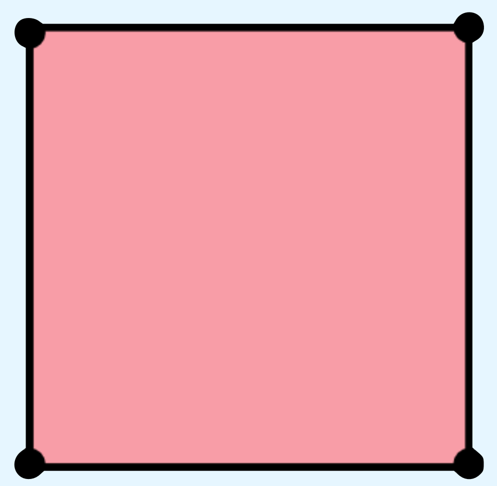

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Bjorken, Landau, Nakanishi ’54

Klausen ’21 - Berghoff, Panzer ’22, - Dlapa, Helmer, Papathanasiou, Tellander ’23

χ-discriminants

$$X_z = \{\alpha \in (\mathbb{C}^*)^n : f_i(\alpha, z) \neq 0, i = 1, \ldots, \ell\}$$

$\mathscr{E} = \mathscr{K}$

$$Z_k(\mathscr{E}) = \{z \in \mathscr{E} : |\chi(X_z)| \leq k\}$$

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$\mathcal{X}$-discriminants

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Definition

The $\mathcal{X}$-discriminant variety of the family X_z of very affine varieties over $\mathscr{E}$ is the closed subvariety

$$\nabla_\chi(\mathscr{E}) = Z_{\chi^*-1}(\mathscr{E}) = \mathscr{E} \backslash V_{\chi^*}(\mathscr{E}) \subset \mathbb{C}^s$$

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Example If $\mathscr{E} = \mathbb{C}^s$ and $\chi^* = \mathrm{vol}(A)$, then $\Delta_\chi(\mathscr{E}) = E_A$

Principal Landau Determinants

$$Y_{G,Q}(\mathscr{E}) \;=\; \Big\{(\alpha, z) \in (\mathbb{C}^*)^n \times \mathscr{E} \;:\; \mathscr{G}_{G,Q}(\alpha; z) = \partial_\alpha \mathscr{G}_{G,Q}(\alpha; z) \;=\; 0\Big\}$$

Principal Landau Determinants

Decompose into distinct, irreducible varieties

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Definition: The principal Landau determinant associated with G and $\mathscr{E}$ is the unique (up to scale) square-free polynomial

$$E_G(\mathscr{E}) = \prod_{Q \in F(A)} \prod_{i \in I(G,Q)_1} \Delta^{(i)}_{G,Q}(\mathscr{E}) \in \mathbb{C}[\mathscr{E}]$$

Symbolic and Numeric Algorithm: PLD.jl in julia on MATHEMATICAL SOFTWARE REPOSITORY

```
edges = [[3,1],[1,2],[2,3],[2,3]];
nodes = [1,1,2,3];
getPLD(edges, nodes, internal_masses = :generic,
                     external_masses = :generic)
```

Symbolic and Numeric Algorithm: PLD.jl in julia on MATHEMATICAL SOFTWARE REPOSITORY

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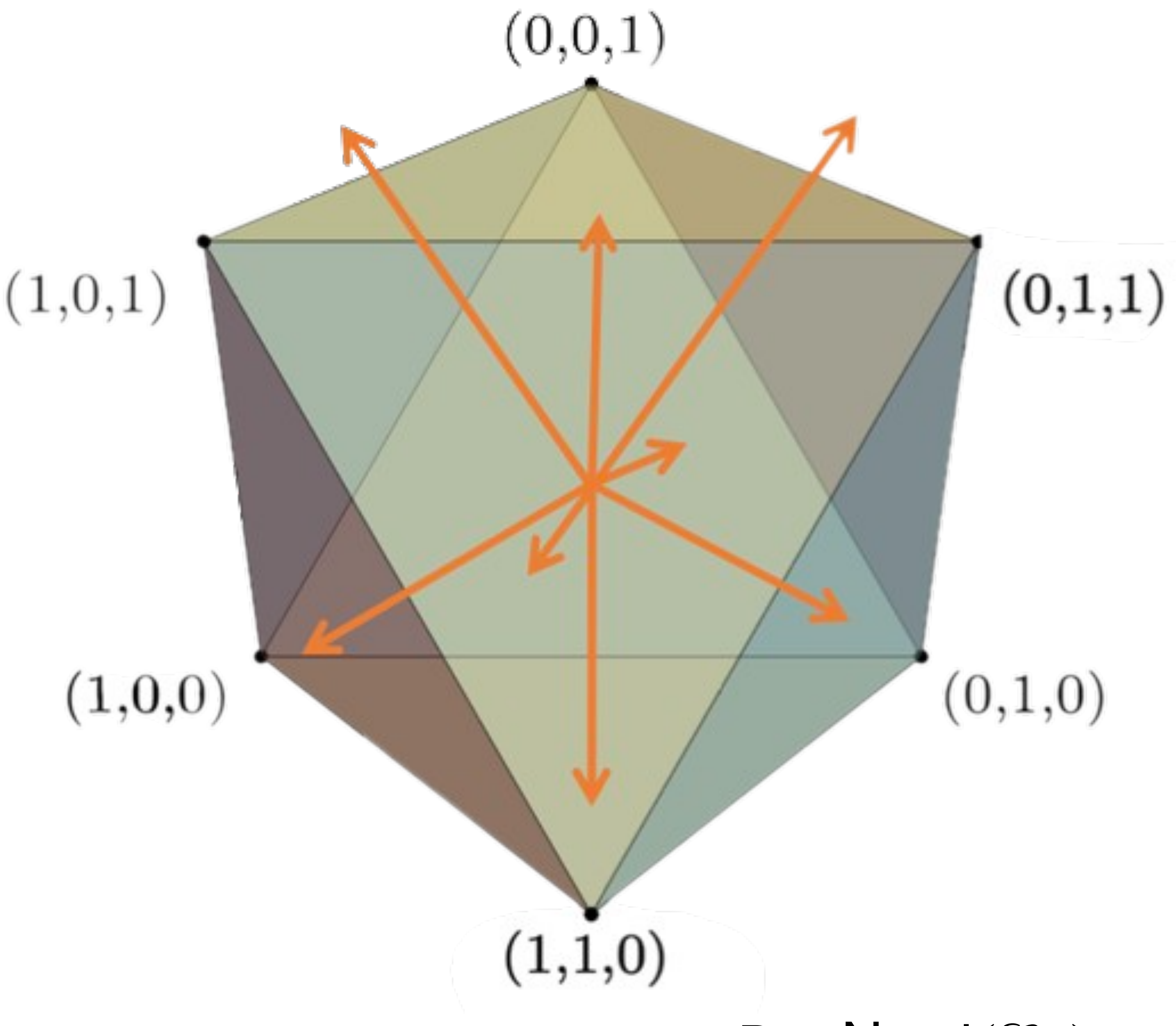

$P = \mathrm{Newt}(\mathscr{G}_G)$

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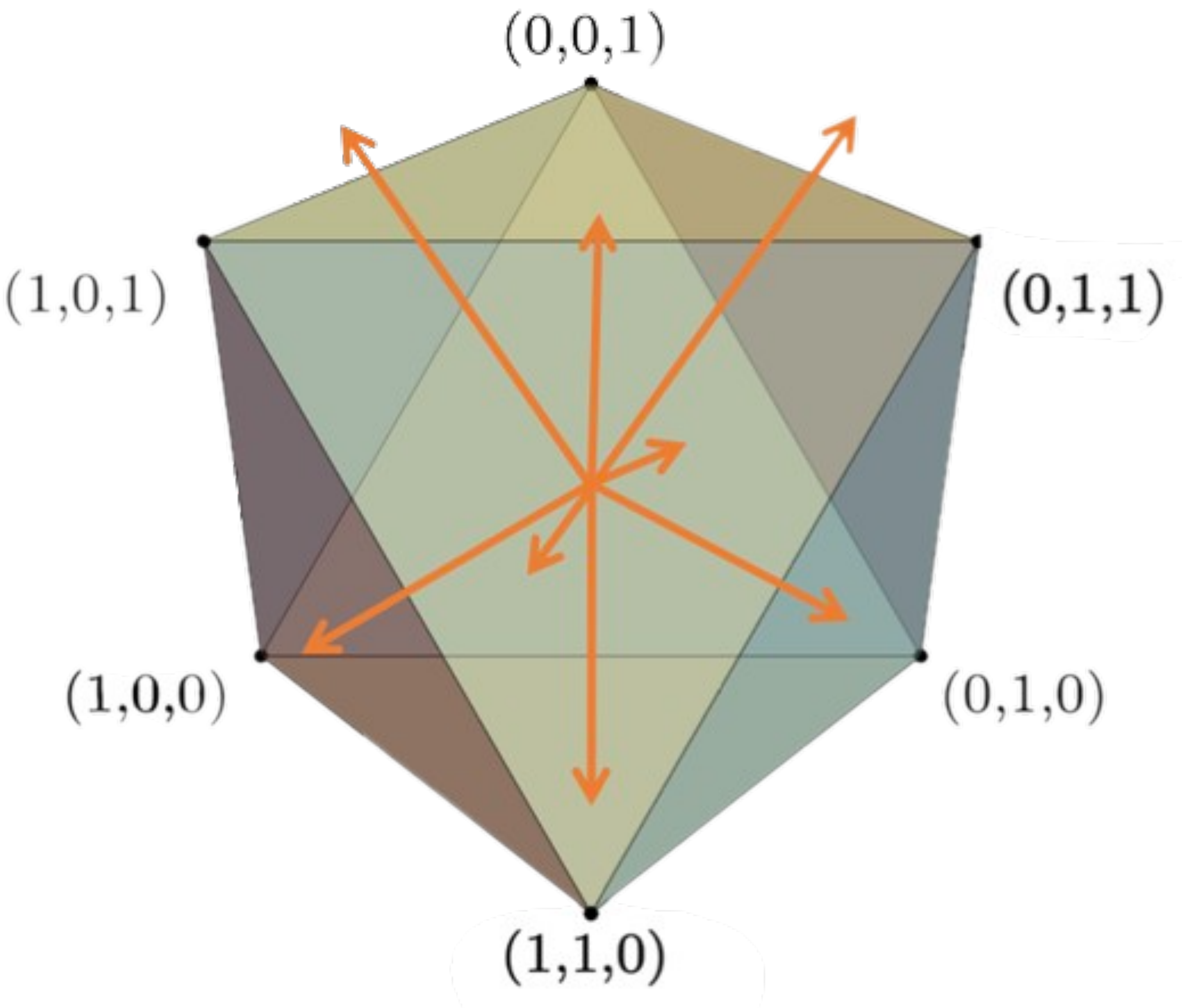

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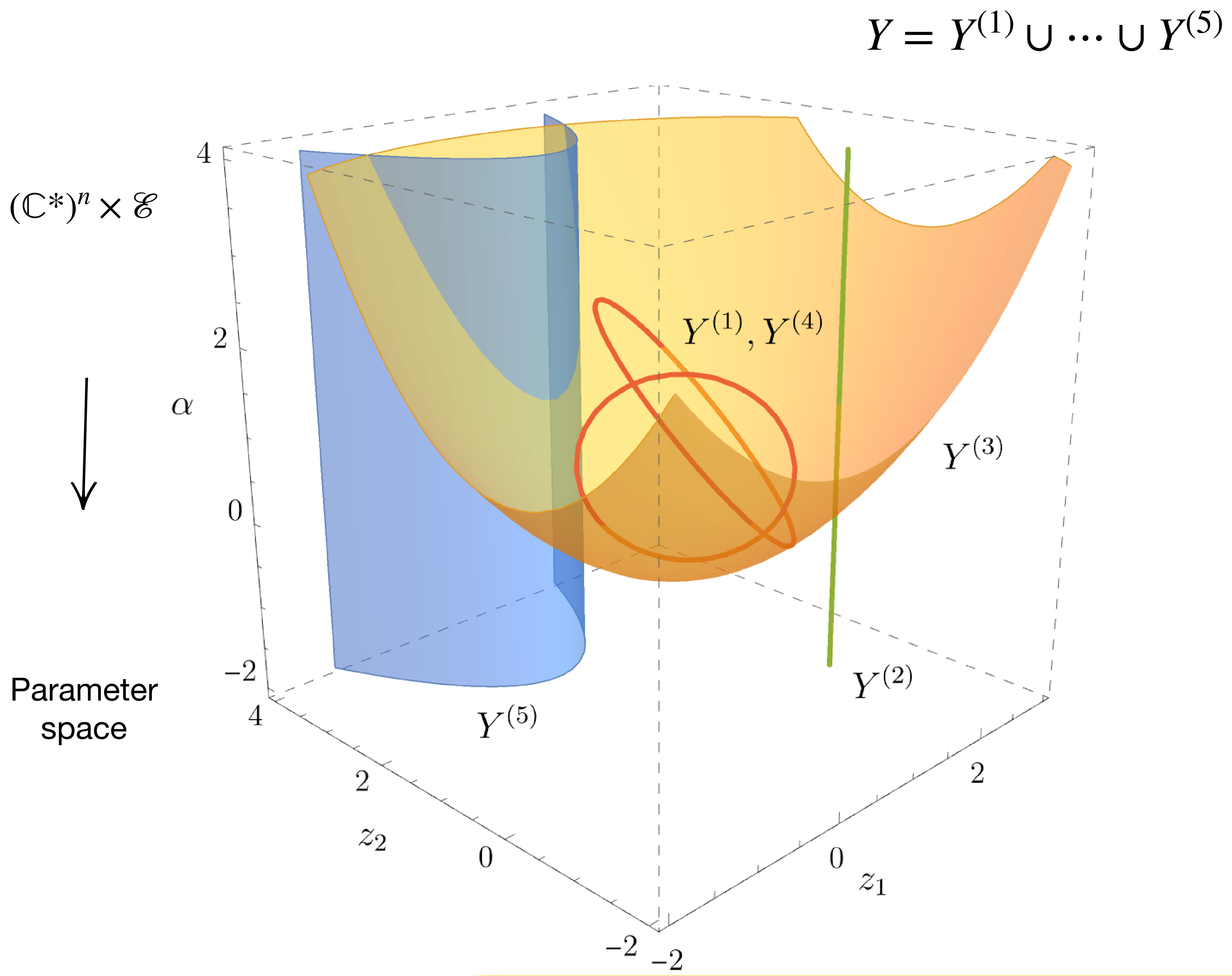

Discard dominant components

Output

Conjectures

$$\text{PLD}_G(\mathscr{E}) \subset \nabla_\chi(\mathscr{E})$$

```
Generic |Euler characteristic|, χ* = 4
candidates = Any[M₁, M₃, M₂, M₁^2 - 2*M₁*M₂ - 2*M₁*M₃ + M₂^2 - 2*M₂*M₃ + M₃^2]
Subspace M₁ has χ = 2 < χ*
Subspace M₃ has χ = 2 < χ*
Subspace M₂ has χ = 2 < χ*
Subspace M₁^2 - 2*M₁*M₂ - 2*M₁*M₃ + M₂^2 - 2*M₂*M₃ + M₃^2 has χ = 3 < χ*
(Any[M₁, M₃, M₂, M₁^2 - 2*M₁*M₂ - 2*M₁*M₃ + M₂^2 - 2*M₂*M₃ + M₃^2], Any[2, 2, 2, 3])
```

But we know $\nabla_\chi(\mathscr{E}) \not\subset \text{PLD}_G(\mathscr{E})$

Conjectures

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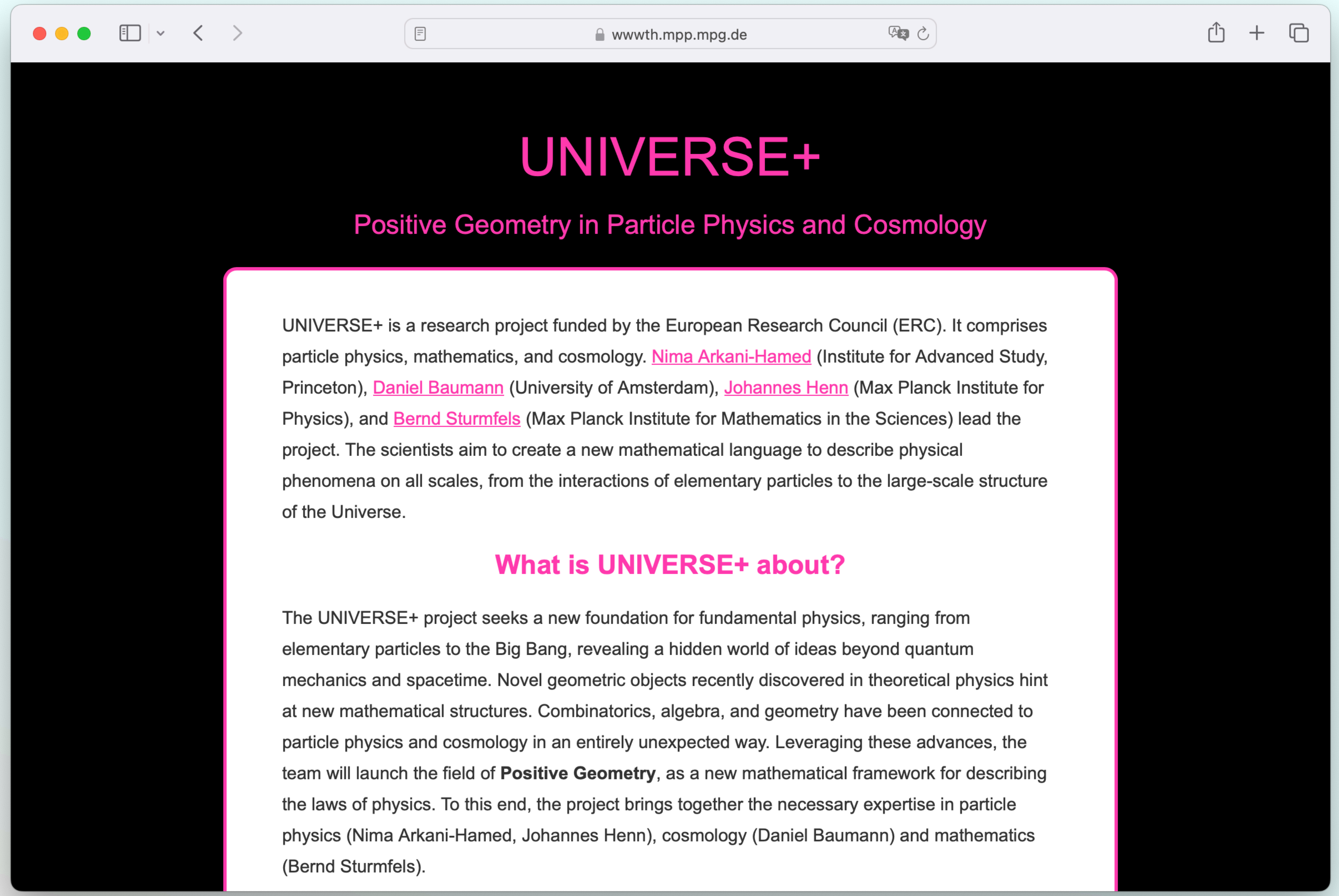
$$\mathrm{PLD}_G(\mathscr{E}) \subset \nabla_\chi(\mathscr{E}) = \mathrm{Sing}(H_A(\kappa, \mathscr{E}))$$

European Research Council
Established by the European Commission


wwwth.mpp.mpg.de

UNIVERSE+

Positive Geometry in Particle Physics and Cosmology

UNIVERSE+ is a research project funded by the European Research Council (ERC). It comprises particle physics, mathematics, and cosmology. Nima Arkani-Hamed (Institute for Advanced Study, Princeton), Daniel Baumann (University of Amsterdam), Johannes Henn (Max Planck Institute for Physics), and Bernd Sturmfels (Max Planck Institute for Mathematics in the Sciences) lead the project. The scientists aim to create a new mathematical language to describe physical phenomena on all scales, from the interactions of elementary particles to the large-scale structure of the Universe.

What is UNIVERSE+ about?

The UNIVERSE+ project seeks a new foundation for fundamental physics, ranging from elementary particles to the Big Bang, revealing a hidden world of ideas beyond quantum mechanics and spacetime. Novel geometric objects recently discovered in theoretical physics hint at new mathematical structures. Combinatorics, algebra, and geometry have been connected to particle physics and cosmology in an entirely unexpected way. Leveraging these advances, the team will launch the field of **Positive Geometry**, as a new mathematical framework for describing the laws of physics. To this end, the project brings together the necessary expertise in particle physics (Nima Arkani-Hamed, Johannes Henn), cosmology (Daniel Baumann) and mathematics (Bernd Sturmfels).

/ CONFERENCE / 12/02/2024 ⇄ 16/02/2024

Positive Geometry in Particle Physics and Cosmology

MPI für Mathematik in den Naturwissenschaften Leipzig

E1 05 (Leibniz-Saal)

Thank you!