Extremal Triangle-Free Graphs

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Outline

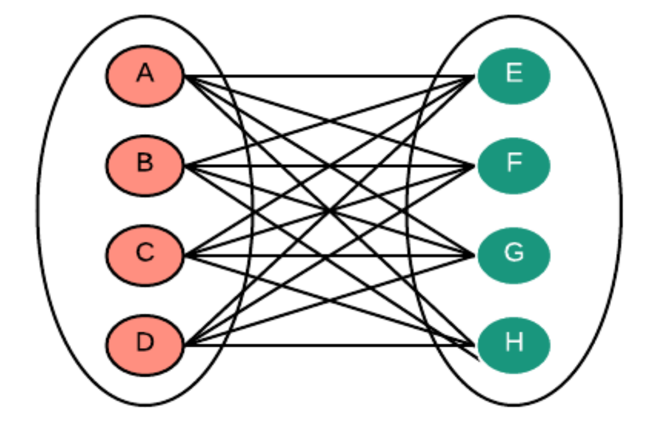
- Results from 2 papers:
  - Milad Ahanjideh, Tınaz Ekim and Mehmet Akif Yıldız, Maximum size of a triangle-free graph with bounded maximum degree and matching number, under revision, 2022, arXiv:2207.02271.
  - Ali Erdem Banak, Tınaz Ekim, Z.Caner Taşkın, Constructing extremal triangle-free graphs using integer programming, Discrete Optimization, 50 (2023), 100802,
- Content:
  - Description of the extremal problem
  - Structural graph theoretical results
  - Integer Programming formulations and Conjectures

Extremal Graph Theory

How large or small can a graph parameter be under a given set of conditions?

Turán's theorem (special case): The max number of edges in a triangle-free graph G with n vertices is $\lfloor n^2/4 \rfloor$; it is achieved by a balanced complete bipartite graph.

$\lfloor n/2 \rfloor$

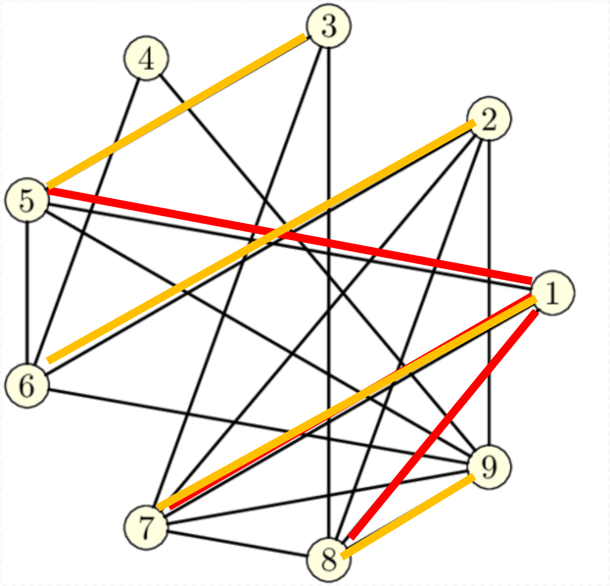


$\lceil n/2 \rceil$

Edge-extremal Problem

How many edges can a graph have at most, under restrictions on its maximum degree and matching number?

max	*number of edges*	**max**	$\mid E(G) \mid$
s.t.	maximum degree ≤ d	**s.t.**	$\Delta(G) \leq d$
	matching number ≤ m		$\nu(G) \leq m$



$d(1)=3$
$d(6)=4$
$\Delta(G)=d(9)=6$

$\nu(G) = 4$

Edge-extremal Problem

A graph of degree at most d and matching number at most m and having a maximum number of edges is called ***(edge-) extremal***.

Q1: Find the opt value

Q2: Find/characterize all extremal graphs

Q3: In which cases the extremal graphs are unique?

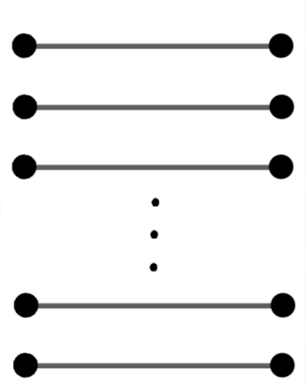
Edge-extremal Problem

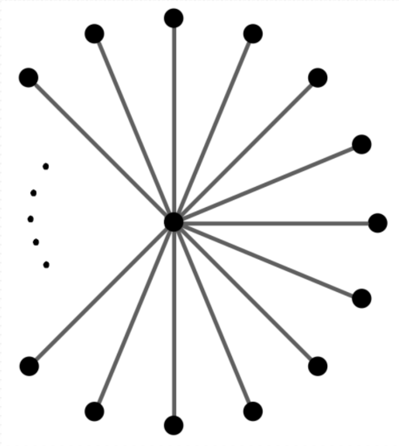
max $|E(G)|$

s.t. $\Delta(G) \leq d$

max $|E(G)|$

s.t. $\nu(G) \leq m$





For $|E(G)|$ to be finite, we need to bound both $\Delta(G)$ and $\nu(G)$.

Literature

- Erdös, Rado, 1960 : a more general problem about set systems
- Chvátal, Hanson, 1976 : answers our question for general graphs and finds some extremal graphs (LP ideas and Berge's matching formula)
- Balachandran, Khare, 2009 : alternative proof with structural approach → construction of extremal graphs (but no information about their uniqueness apart from a few cases)

Edge-extremal General Graphs

$\mathbb{M}_{\mathbf{C}}(d,m)$: All graphs G in the class **C** s.t. $\Delta(G) \leq d$ and $\nu(G) \leq m$

$f_{\mathbf{C}}(d,m)$: maximum number of edges in a graph in $\mathbb{M}_{\mathbf{C}}(d,m)$

: size of an edge-extremal graph in C (for d and m)

Vizing's Theorem: $\chi'(G) \leq \Delta(G)+1$

$$|E(G)| \leq (\Delta(G)+1)\nu(G) \leq dm + m$$

This upper bound is met when some divisibility conditions for d and m hold, and we are pretty close otherwise.

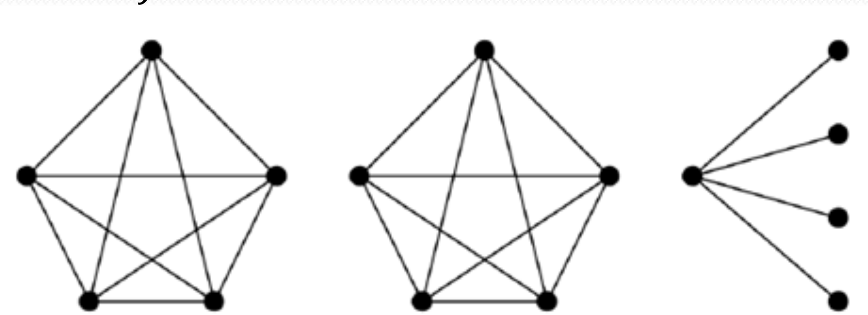
Edge-extremal General Graphs

$\mathcal{GEN}$: Class of general graphs

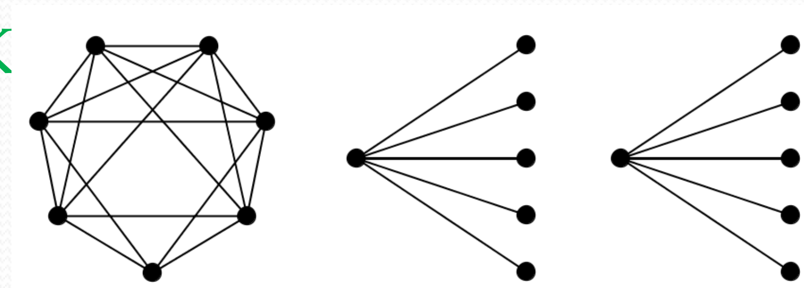
Thm (Balachandran, Khare, 2009): $f_{\mathcal{GEN}}(d,m) = dm + \left\lfloor \frac{d}{2} \right\rfloor \left\lfloor \frac{m}{\lceil \frac{d}{2} \rceil} \right\rfloor$

Moreover, an edge-extremal graph can be obtained by taking
disjoint union of d-stars, complete graphs K_{d+1} (if d even)

s K

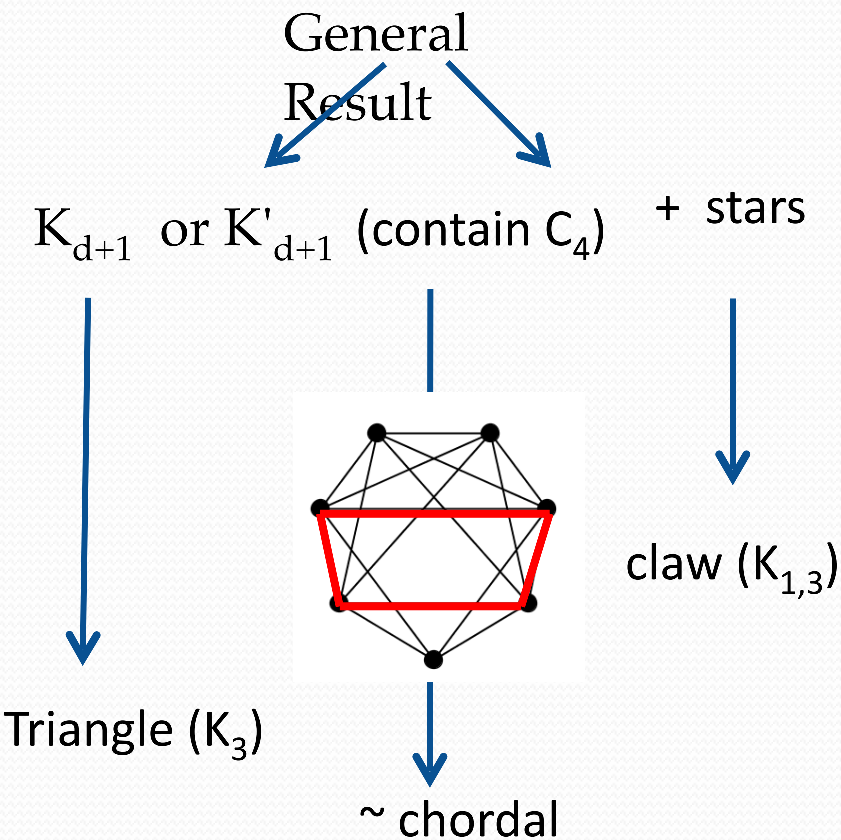


An edge-extremal graph for d=4, m=5



An edge-extremal graph for d=5, m=5

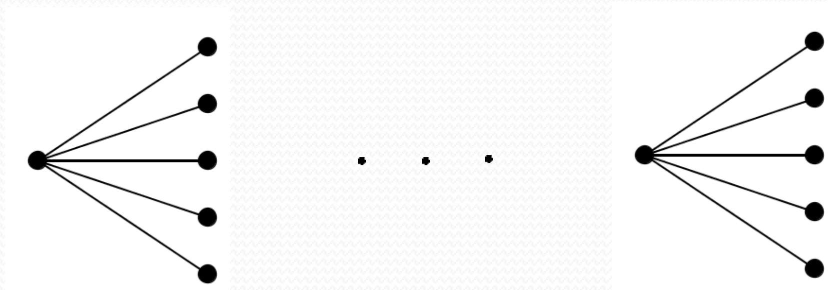
What if we restrict the structure of extremal graphs? Can the same upper bound be still achieved?



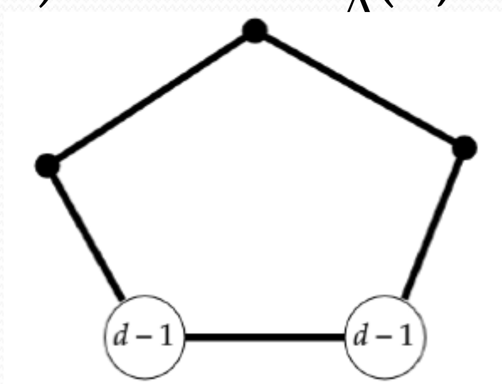
- (Dibek, E., Heggernes, 2017): d small enough ➔ Max |E(G)| remains the same; d large enough ➔ max|E(G)| decreases
- (Blair, Heggernes, Lima, Lokshtanov, 2020) (more restricted than C_4-free graphs): the general bound not achieved.
- (Maland, 2015): Bipartite graphs, split graphs, unit interval graphs
- Open question: T-free graphs? $f_\Delta(d, m) = ?$

Edge-extremal T-free graphs with d≥m

- Thm: $f_\Delta(d, m) = dm$ for $d > m \geq 1$. (m many d-stars)



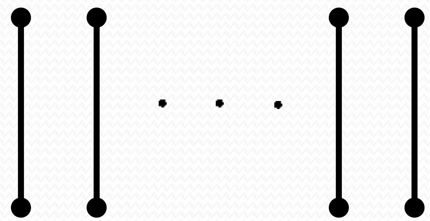
- Thm: $f_\Delta(1, 1) = 1$ and $f_\Lambda(d, d) = d^2 + 1$ for $d \geq 2$.



A_d : Blow-up of C_5

Edge-extremal T-free graphs with d<m

- Thm: $f_\Delta(1, m)=m$ for all $m \geq 1$. (m independent edges)



→ Assume $d \geq 2$

- Key Result: Let G be an edge-extremal T-free graph with $\Delta(G) \leq d$, $\nu(G) \leq m$ having a maximum number of d-star components. Then every component H of G that is not a d-star
  - factor-critical (thus $|V(H)| = 2\nu(H)+1$)
  - with $d \leq \nu(H) \leq Z(d)$, and
  - edge-extremal for d and $\nu(H)$ ($|E(H)|=f_\Delta(d, \nu(H))$)

Edge-extremal T-free graphs with d<m

- Def: For any d ≥2, let **Z(d)** be the smallest natural number n s.t. there exists a d-regular (if d is even) or almost d-regular (if d is odd) T-free and factor-critical graph G with ν(G) = n.
- Lemma: For every d ≥ 2, the value Z(d) and a T-free factor-critical (almost) d-regular graph $\mathbf{B_d}$ with matching number Z(d) exist.
- The value of Z(d) is crucial:
  - Z(d) known for d ≤ 6;
  - $Z(d) = \lfloor 5d/4 \rfloor$ if d ≥ 2 even;
  - $Z(d) = \lfloor 5(d-1)/4 \rfloor \leq Z(d) \leq \lfloor 5(d+1)/4 \rfloor$ if d ≥ 2 odd.

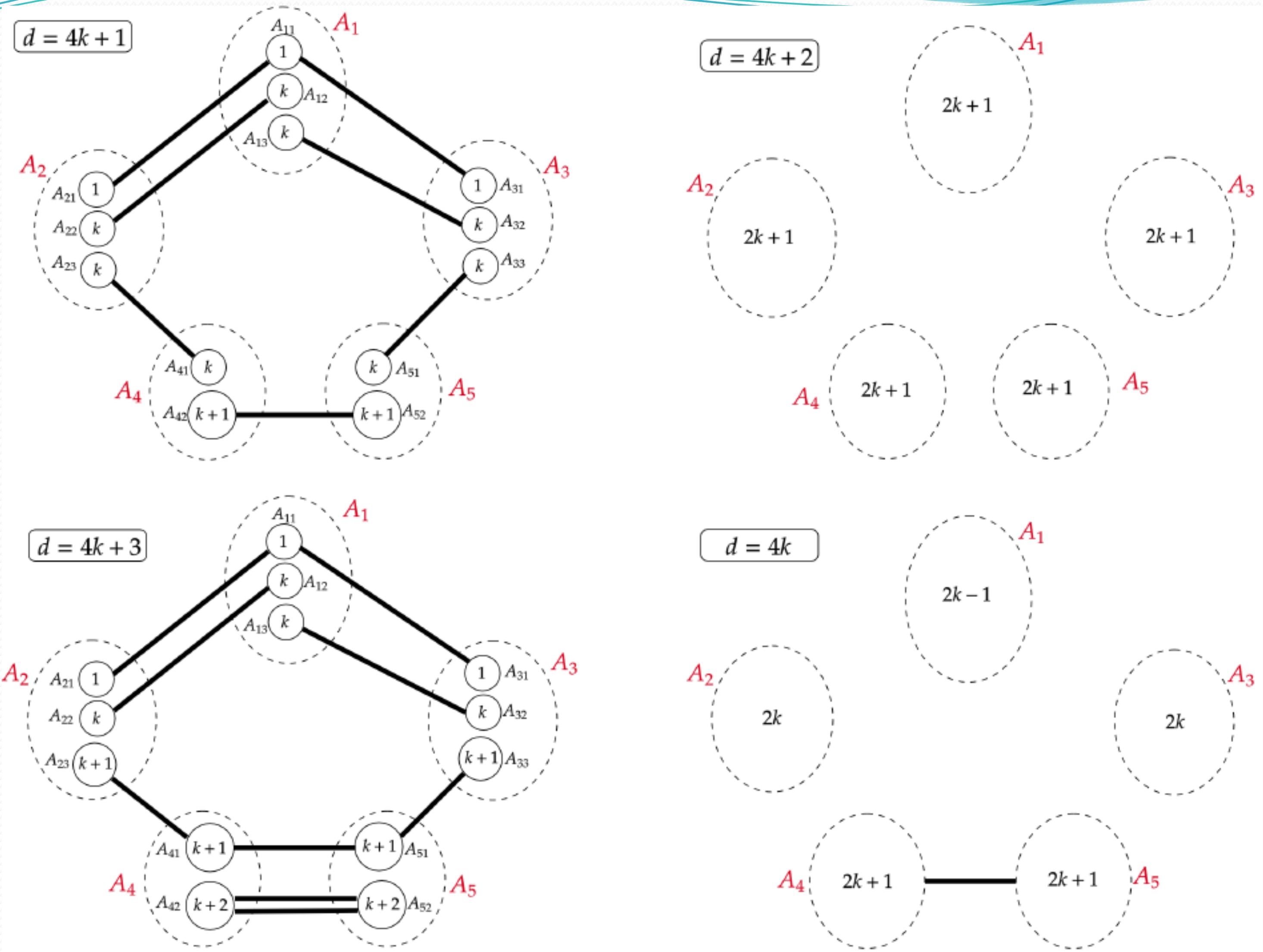
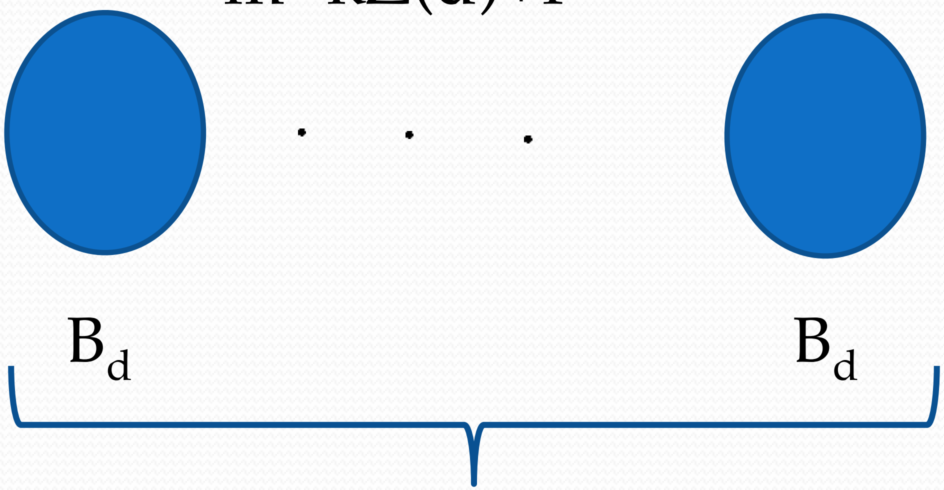


Figure 2: The graph B_d for $d \geq 2$ depending on d (mod 4).

Edge-extremal T-free graphs with d<m and Z(d) ≤ m < 2d

Thm (idea): Take as many B_d as possible, complete the matching number with d-stars → $dm + \lfloor d/2 \rfloor$ edges

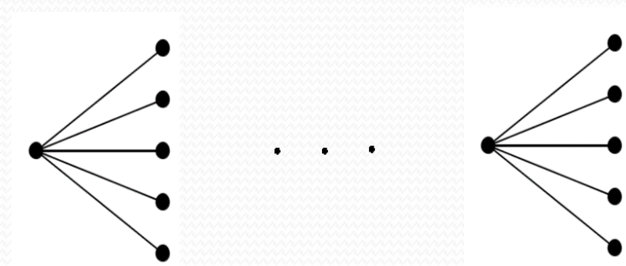
$m = kZ(d) + r$



B_d B_d

Matching number = k Z(d)

edges = (2Z(d)+1) d/2



r many d-stars

matching number = r

edges = (m-Z(d))d

Main Theorem

Let d and m be natural numbers with $d \geq 2$, let k and r be nonnegative integers such that $m = kZ(d) + r$ with $0 \leq r < Z(d)$. Then, for all the cases with

- $d \geq m$
- $d < m$ and $d \leq 6$
- $d < m$ and $Z(d) \leq m < 2d$

we have,

$$f_\triangle(d,m) = \begin{cases} dm + k\lfloor d/2 \rfloor & \text{if } r < d, \\ dm + k\lfloor d/2 \rfloor + r - d + 1 & \text{if } r \geq d, \end{cases}$$

where an extremal graph can be constructed as the disjoint union of k copies of B_d and

(i) A_d if $r \geq d$,

(ii) r copies of d-stars if $r < d$.

Deviation from general extremal graphs

$$h_{\triangle}(d,m) = \begin{cases} 0, & \text{if } m < \lfloor d/2 \rfloor, \\ \lfloor d/2 \rfloor, & \text{if } \lfloor d/2 \rfloor \le m < d \\ d, & \text{if } d = m \text{ and } d \text{ is even}, \\ \lfloor d/2 \rfloor, & \text{if } d = m \text{ and } d \text{ is odd}, \\ 0, & \text{if } 1 = d < m, \\ m - \lfloor m/2 \rfloor, & \text{if } 2 = d < m, \\ \lfloor m/2 \rfloor - \lfloor m/3 \rfloor, & \text{if } 3 = d < m, \\ 2\lfloor m/2 \rfloor - 2\lfloor m/5 \rfloor, & \text{if } 4 = d < m \text{ and } m+1 \text{ is not divisible by } 5, \\ 2\lfloor m/2 \rfloor - 2\lfloor m/5 \rfloor - 1, & \text{if } 4 = d < m \text{ and } m+1 \text{ is divisible by } 5, \\ 2\lfloor m/3 \rfloor - 2\lfloor m/6 \rfloor, & \text{if } 5 = d < m \text{ and } m+1 \text{ is not divisible by } 6, \\ 2\lfloor m/3 \rfloor - 2\lfloor m/6 \rfloor - 1, & \text{if } 5 = d < m \text{ and } m+1 \text{ is divisible by } 6, \\ 3\lfloor m/3 \rfloor - 3\lfloor m/7 \rfloor, & \text{if } 6 = d < m \text{ and } m+1 \text{ is not divisible by } 7, \\ 3\lfloor m/3 \rfloor - 3\lfloor m/7 \rfloor - 1, & \text{if } 6 = d < m \text{ and } m+1 \text{ is divisible by } 7, \\ \lfloor d/2 \rfloor, & \text{if } d \ge 7 \text{ and } Z(d) \le m < 3\lceil d/2 \rceil, \\ 2\lfloor d/2 \rfloor, & \text{if } d \ge 7 \text{ and } 3\lceil d/2 \rceil \le m < 2d. \end{cases}$$

Integer Programming formulation

- Open cases: $7 \leq d < m$ and either $m < Z(d)$ or $m \geq 2d$
- Reminder (Key result): Every component H of an edge-extremal graph G that is not a d-star is edge-extremal T-free graph for d and $\nu(H)$ where $d \leq \nu(H) \leq Z(d)$ ➔ extremal components
- Decision variable x_i = # of extremal components of G whose matching number is i where $d \leq i \leq Z(d)$.
- How many of each extremal component with matching number i (x_i) for each $d \leq i \leq Z(d)$?
- How many d-stars?

Knapsack formulation

$$\sum_{i=d}^{Z(d)} i x_i \leq m$$

$$f_\Delta(d,m) = d\left(m - \sum_{i=d}^{Z(d)} i x_i\right) + \sum_{i=d}^{Z(d)} f_\Delta(d,i) x_i$$

Extremal components

$$\max\ dm + \sum_{i=d}^{Z(d)} (f_\Delta(d,i) - di) x_i$$

utilities

$$m - \sum_{i=d}^{Z(d)} i x_i$$

$$\text{subject to} \sum_{i=d}^{Z(d)} i x_i \leq m$$

volumes

d-stars

$$x_i \geq 0, x_i \in \mathbb{Z}$$

Efficiently solved

Conjectures for open cases

- Conj 1: Main thm also holds for $6<d<m<Z(d)$; for i s.t. $6<d<i<Z(d)$ we have $f_\Delta(d,i)=di+i-d+1$.
- Prop: If Conj 1 true then (using $f_\Delta(d,Z(d))= dZ(d)+\lfloor d/2\rfloor$) IP equivalent to the following formulation which admits an opt. sol. where $x_{Z(d)}$ is maximized and there is at most one other extremal component with smaller matching number.

$$\max \lfloor d/2\rfloor x_{Z(d)} + \sum_{i=d}^{Z(d)-1} (i-d+1)x_i$$

$$\text{subject to } \sum_{i=d}^{Z(d)} i x_i \leq m$$

$$x_i \geq 0, x_i \in \mathbb{Z}$$

Interpretation→ New conjecture

- If Conj 1 holds, $f_\Delta(d,i)$ edges can be achieved by taking the graph B_d as much as possible and adding either one extremal component for d and r; or r many d-stars, depending on r ≥ d where r = remainder of m when divided by Z(d). (Just like in the main Thm)
- Conj 2: . *Let* $m = kZ(d) + r$ *for some* $0 \le r < Z(d)$. *Then, we have*

$$f_\Delta(d,m) = \begin{cases} dm + k\lfloor d/2 \rfloor & \text{if } r < d \\ dm + k\lfloor d/2 \rfloor + r - d + 1 & \text{if } r \ge d. \end{cases}$$

- Conj 3: For d≥21 and odd, we have $Z(d) = \lfloor 5(d+1)/4 \rfloor$.

New IP for extremal components $f_\Delta(d,i)$

- IP formulations for constructing extremal components for i s.t. $6<d<i<Z(d)$ ➔ missing parameters for Knapsack
- Decision var. x_{ij} = edge ij exists or not

Max $\sum x_{ij}$

s.t. T-free

$\Delta(G) \leq d$ — Key Lemma

$\nu(G) \leq m$

x_{ij} binary

$$\max \sum_{i,j \in V} x_{ij}$$

$$\text{s.t.} \quad x_{ij} + x_{jk} + x_{ik} \leq 2 \quad \forall i,j,k \in V$$

$$\sum_{j \in V} x_{ij} \leq d \quad \forall i \in V$$

$$x_{ij} \in \{0,1\} \quad \forall i,j \in V$$

👍 Using IP for constructing graphs with structure

👎 High symmetry!

Methodology

1. Basic Formulation + CPLEX symmetry breaking 5
2. Orbital Branching to eliminate isomorphisms in the branch&cut tree: find symmetry / automorphism groups -> variable orbits and constraint orbits (equivalence classes). Reducing symmetry
3. Iterative formulation: Set all degrees to d, then all but one to d, etc. searching for a feasible solution, until UB=LB. Reducing the feasible region.
4. Combination of Orbital Branching and Iterative Approach

Computational experiments

- Intel(R) Core(TM) i7-9750H CPU @ 2.60GHz with 32 GB RAM using CPLEX 20.1.0 on 10 threads.
- CPLEX 20.1.0 with C++ to implement the methods
- A limit of 1800 seconds for each run
- Orbital Branching and Iterative Method with callback mechanism of CPLEX. Orbits found by nauty 2.7r3 software library to compute automorphism groups [McKay, Piperno, 2014]

Main findings with IP

- We construct all extremal components for d=7,8,9,10 and $d<m<Z(d)$ with IP + some for d=11,12 and 13.
- We solve the Knapsack Formulation for d=7,8,9,10 and any m value
- All our findings support Conjectures 1, 2.
- All extremal graphs have a max number of $x_{Z(d)}$ and at most one other extremal component.
- As a byproduct (checking the degrees), we obtain new values for Z(d): Z(7)=9, Z(9)=13, Z(11)=15, and Z(13)=17. (By definition of Z(d) and checking the degrees)

Best computational results

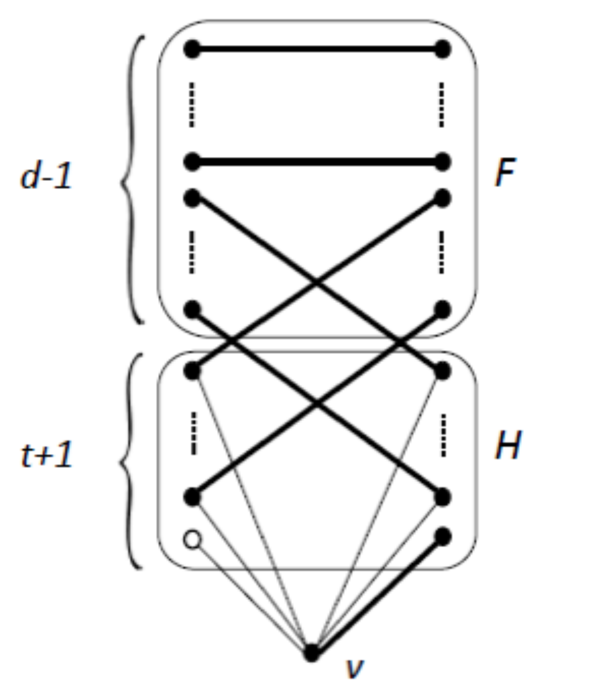
Parameters			Iterative + Orbital Branching				
d	m	PreUB	LB	UB	Gap	Time	Node
7	8	59	58	58	0.00%	0	57
7	9	66	66	66	0.00%	0	0
8	9	76	74	74	0.00%	4	2794
8	10	84	84	84	0.00%	0	0
9	10	94	92	92	0.00%	11	1990
9	11	103	102	102	0.00%	36	4170
9	12	112	112	112	0.00%	368	229
10	11	115	112	112	0.00%	79	10114
10	12	125	125	125	0.00%	2	0
11	12	137	134	134	0.00%	146	15106
11	13	148	146	146	0.00%	355	371215
11	14	159	158	159	0.63%	1800	706832
11	15	170	170	170	0.00%	1	0
12	13	162	158	158	0.00%	1623	2020924
12	14	174	171	172	0.58%	1800	1101510
12	15	186	186	186	0.00%	0	0
13	14	188	184	185	0.54%	1800	2294463
13	15	201	198	200	1.01%	1800	935875
13	16	214	212	213	0.47%	1800	577438
13	17	227	227	227	0.00%	72	543
Avg			138.45	138.75	0.22%	585	402163

Bottlenecks:

1) Computing the orbits
2) Decreasing the UB

Pattern for extremal components suggested by IP

- Prop: For all $i = d + t$ such that $d < i < Z(d)$, the graph $B_{d,d+t}$ is an extremal graph if Conjecture 1 holds (with $di+i-d+1$ edges, $\Delta(B_{d,d+t}) = d$ and $\nu(B_{d,d+t}) = d + t = i$)



- H independent set
- F complete bipartite – t distinct perfect matching
- Opposite parts of F and H are completely linked

Generalizes the extremal graph A_d for $f_\Delta(d, d)$

Knapsack formulation

d	m	edge count	$\#d$-star	$\#Z(d)$	#other
7	15	108	6	1	0
7	16	116	0	1	1
7	17	124	0	1	1
7	18	132	0	2	0
7	19	139	1	2	0
7	20	146	2	2	0
7	21	153	3	2	0
8	17	140	7	1	0
8	18	148	8	1	0
8	19	158	0	1	1
8	20	168	0	2	0
8	21	176	1	2	0
8	22	184	2	2	0
8	23	192	3	2	0
8	24	200	4	2	0
9	19	175	7	1	0
9	20	184	8	1	0
9	21	194	0	1	1
9	22	204	0	1	1
9	23	214	0	1	1
9	24	224	0	2	0
9	25	233	1	2	0
9	26	242	2	2	0
9	27	251	3	2	0
10	21	215	9	1	0
10	22	226	0	1	1
10	23	237	0	1	1
10	24	250	0	2	0
10	25	260	1	2	0
10	26	270	2	2	0
10	27	280	3	2	0
10	28	290	4	2	0
10	29	300	5	2	0
10	30	310	6	2	0

No practical limit: d = 10 and m = 10000 takes 0.2 sec.

$2d < m \leq 3d$

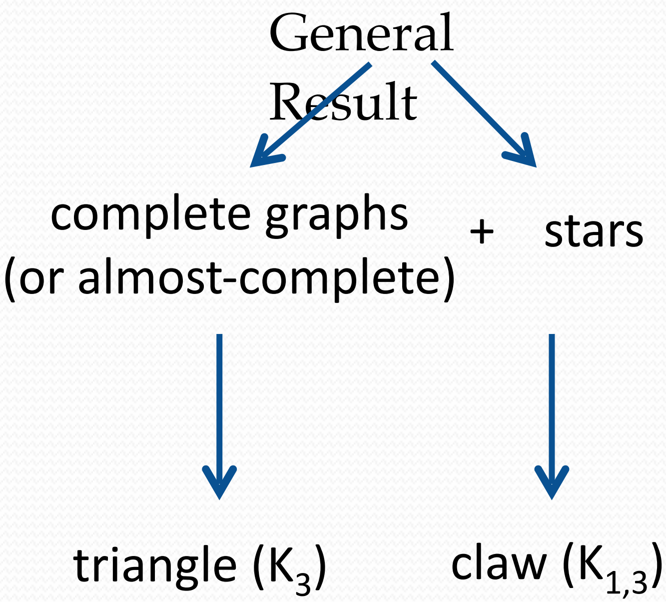
d=8 and $m \geq 2d$

d	m	edge count	d-star	comp_8	comp_9	comp_10
8	15	124	5	0	0	1
8	16	132	6	0	0	1
8	17	140	7	0	0	1
8	18	149	0	1	0	1
8	19	158	0	0	1	1
8	20	168	0	0	0	2
8	21	176	1	0	0	2
8	22	184	2	0	0	2
8	23	192	3	0	0	2
8	24	200	4	0	0	2
8	25	208	5	0	0	2
8	26	216	6	0	0	2
8	27	224	7	0	0	2
8	28	233	0	1	0	2
8	29	242	0	0	1	2
8	30	252	0	0	0	3
8	31	260	1	0	0	3
8	32	268	2	0	0	3
8	33	276	3	0	0	3
8	34	284	4	0	0	3
8	35	292	5	0	0	3
8	36	300	6	0	0	3
8	37	308	7	0	0	3
8	38	317	0	1	0	3
8	39	326	0	0	1	3
8	40	336	0	0	0	4
8	41	344	1	0	0	4
8	42	352	2	0	0	4
8	43	360	3	0	0	4
8	44	368	4	0	0	4
8	45	376	5	0	0	4
8	46	384	6	0	0	4
8	47	392	7	0	0	4

Conclusion

- IP methods
  - shed some light on the open cases
  - provides more motivation to search for a formal proof
  - but they can never settle all open cases
- Structural proofs for the conjectures require more powerful techniques
- Nice combination of IP and structural graph theory
- Opens the way to use IP to construct graphs with desired properties

Future Work



- What if we forbid k-star? (k≥4) Can we find an explicit formula in terms of d, m, and k, for the maximum number of edges that a graph G can have where $\Delta(G) \leq d$, $\nu(G) \leq m$ and where G does not contain k-star as an induced subgraph?
- What if we forbid k-clique? What is the maximum number of edges of a graph G where $\Delta(G) \leq d$, $\nu(G) \leq m$ and $\omega(G)<k$?
- What if we require connectivity for general edge-extremal graphs?

Thank you for listening

A question inspired by IP

- IP for extremal components: $|V(H)|=2\,\nu(H)+1$ but we do not explicitly force that H is factor-critical.
- However, it turns out that all extremal components resulting from IP are factor-critical.

$$\max \sum_{i,j\in V} x_{ij}$$

$$\text{s.t.} \quad x_{ij} + x_{jk} + x_{ik} \leq 2 \quad \forall i,j,k \in V$$

$$\sum_{j\in V} x_{ij} \leq d \quad \forall i \in V$$

$$x_{ij} \in \{0,1\} \quad \forall i,j \in V$$

QUESTION: Is it true that a triangle-free extremal graph G with matching number m and maximum degree d such that $d < m < Z(d)$ and having $2m + 1$ vertices is factor-critical?

Relation to Ramsey Numbers

Observation: Let *G* be a graph, let *L(G)* be the line graph of *G*, and let $d \geq 4$ and $j \geq 1$ be two integers. Then *G* has a vertex of degree at least *d* if and only if *L(G)* has a clique of size *d*. Moreover, *G* has a matching of size m if and only if *L(G)* has an independent set of size *m*.

Graph G		Graph L(G)	
max	*number of edges*	**max**	*number of vertices*
s.t.	maximum degree ≤ d	**s.t.**	clique number ≤ d
	matching number ≤ m		independence number ≤ m

Max number of edges in G = R(d+1,m+1) - 1 for L(G)