Homological mirror symmetry for chain type polynomials

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joint work with Alexander Polishchuk

Chain type polynomials

- For an n-tuple of positive integers $a = (a_1, .., a_n) \in \mathbb{Z}^n_{>1}$, $n \geq 1$, we define the chain polynomial:

$$\boxed{p_a(z_1, \ldots, z_n) := \sum_{i=1}^{n-1} z_i^{a_i} z_{i+1} + z_n^{a_n}}$$

- Isolated singularity at the origin for $p_a : \mathbb{C}^n \to \mathbb{C}$ (tame map)
- The group of (diagonal) symmetries up to scaling of p_a

$$\Gamma_a := \{\lambda_1^{a_1}\lambda_2 = \ldots = \lambda_{n-1}^{a_{n-1}}\lambda_n = \lambda_n^{a_n} = \lambda\} \subset (\mathbb{C}^*)^{n+1}$$

  Group of symmetries $\Gamma_a^0 \subset \Gamma_a$ given by $\lambda = 1$
- Using the Γ_a action on $\mathbb{C}$ by multiplication with λ, $p_a : \mathbb{C}^n \to \mathbb{C}$ becomes Γ_a-equivariant.

Berglund-Hubsch-Henningson duality for chain polynomials

- Define $a^\vee = (a_n, \ldots, a_1)$
- B-Hu: p_a and $p_{a^\vee}$ are "mirror" LG models?
- B-He: Non-trivial check + clarified role of the symmetry groups
- Takahashi: there should exist a triangulated equivalence

$$\boxed{DFuk(p_a) \simeq HMF_{L_{a^\vee}}(p_{a^\vee})}$$

- $L_{a^\vee} := Hom(\Gamma_{a^\vee}, \mathbb{C}^*)$ grading group, acts on the RHS
- Canonical short exact sequence

$$0 \to \mathbb{Z} \to L_{a^\vee} \to Hom(\Gamma^0_{a^\vee}, \mathbb{C}^*) \underset{can.}{\simeq} \Gamma^0_a \to 0.$$

Equivariant HMS conjecture

- Γ_a^0 acts on $\mathbb{C}^n$ by symplectomorphisms and preserves p_a by definition. Taking graded lifts we obtain an action of $L_{a^\vee}$ on $D^b Fuk(p_a)$.
- Equivariant conjecture: there is an HMS equivalence intertwining these actions
- Would this imply the statements with $G \subset \Gamma_a^0$?
- There exists $F \in L_{a^\vee}$ whose action on either side equals T^2, where T is the shift functor
- There exists $P \in L_{a^\vee}$ whose action on either side satisfies

$$P^{\mu(a)} = T^{m(a)} S^{-1}, \tag{1}$$

  where S is the Serre functor, $\mu(a)$, $m(a)$ explicit integers
- $L_{a^\vee} = < F, P \mid d(a)P = \mu(-a)F >$ - gen. by T^2 and S if $(d(a), \mu(a)) = 1$, so equivariance is automatic in that case

Exceptional collections: A-side I

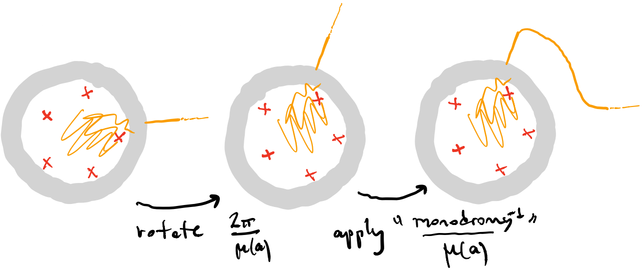
- Milnor number of the singularity of p_a:

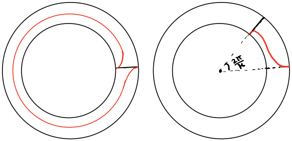
$$\boxed{\mu(a) := a_1 \dots a_n - a_2 \dots a_n + \dots + (-1)^{n-1} a_n + (-1)^n.}$$

- Let $d(a) := a_1 \dots a_n$ and $-a := (a_2, \dots, a_n)$.
- Useful: $\mu(a) + \mu(-a) = d(a)$ and $\Gamma_a \simeq \{\lambda_1^{d(a)} = \lambda^{\mu(-a)}\}$.
- Morsification: $\epsilon z_1 + p_a(z_1, \dots, z_n) = \epsilon z_1 + \sum_{i=1}^{n-1} z_i^{a_i} z_{i+1} + z_n^{a_n}$
- $\epsilon z_1 + p_a : \mathbb{C}^n \to \mathbb{C}$ tame for all $\epsilon \in \mathbb{C}$
- 0 is a regular value for $\epsilon \neq 0$.
- The order $\mu(a)$ cyclic subgroup of Γ_a given by $\lambda_1 = \lambda$ makes $z_1 + p_a$ equivariant (maximal possible for such perturbation)

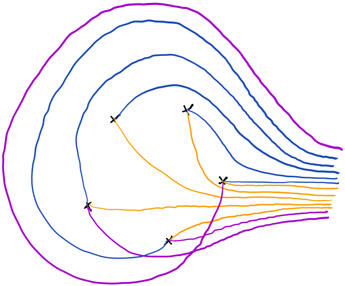
Exceptional collections: A-side II

- $D(F(p_a)) \simeq D(F(\epsilon z_1 + p_a))$ ("same at infinity")
- We can geometrically reconstruct the action of P on $D(F(\epsilon z_1 + p_a))$. On objects up to grading:





Exceptional collections: A-side III



- Blue paths give our distinguished basis of thimbles.
- Yellow ones give the geometric left dual distinguished basis. Purple ones are in the geometric helix of the blue distigiushed basis (up to grading).
- Blue and purple thimbles are generated by a single thimble using $P^{\pm 1}$ (up to grading)

Exceptional collections: A-side IV

- Let us denote the directed Fukaya-Seidel A_∞-category with an exceptional collection corresponding to the blue paths by

$$\boxed{(\mathcal{A}_a, e_a)}$$

- $D(\mathcal{A}_a) \simeq D(F(\epsilon z_1 + p_a))$ — work in progress, but should follow from GPS
- In previous work, I computed the Euler pairing in $K_0(\mathcal{A}_a)$, which is the same as the Seifert form of p_a, with respect to the basis corresponding to e_a. This confirms conjecture by Orlik-Randell '77.
- Set $(\mathcal{A}_\emptyset, e_\emptyset)$ to be the A_∞ cat. with one object $\cdot$ with $Hom^*(\cdot, \cdot) = k$
- We prove that $(\mathcal{A}_a, e_a)$ can be obtained from $(\mathcal{A}_{-a}, e_{-a})$ by an explicit recursive procedure $\mathcal{R}$ for $a \neq \emptyset$

Exceptional collections: B-side

- Aramaki-Takahashi consider the following graded matrix factorization of p_a:

$$E := \begin{cases} \mathrm{stab}(x_2, x_4, \ldots, x_n), & n \text{ even} \\ \mathrm{stab}(x_1, x_3, \ldots, x_n), & n \text{ odd} \end{cases}$$

- The collection $(E, P(E), \ldots, P^{\mu(a^\vee)-1}(E))$ is a full exceptional collection in $HMF_{L_a}(p_a)$.
- AT compute the Euler pairing wrt this basis. My computation for p_a and their computation for $p_{a^\vee}$ give exactly the same result.
- Let us denote their subcategory in an enhancement by

$$\boxed{(\mathcal{B}_a, e_a)}$$

- We prove that $(\mathcal{B}_a, e_a)$ can be obtained from $(\mathcal{B}_{a-}, e_{a-})$ by $\mathcal{R}$

Recursion $\mathcal{R}$

- Start with a directed A_∞-category $(\mathcal{C}, e)$.
- Extend e to a helix inside $Tw(\mathcal{C})$ and take the segment f of length N in this helix such that e is the rightmost subsegment of f.
- Note that f is no longer an exceptional collection in general. We define $\mathcal{C}'$ as the directed A_∞-category defined by the directed A_∞-subcategory of f (keeping track of only morphisms from left to right in the order of the helix).
- Inside $Tw(\mathcal{C}')$, we consider the right dual exceptional collection f^+.
- We say $(\mathcal{C}^+, e^+)$ is obtained from $(\mathcal{C}, e)$ by $\mathcal{R}$ if one can take shifts of the objects of f^+ and find an A_∞-quasi-isomorphism from $(\mathcal{C}^+, e^+)$ to their directed A_∞-subcategory.

Explanations and remarks on recursion $\mathcal{R}$

- Mutations of exceptional collections in a triangulated A_∞-category:

$$E_1, \ldots E_n \mapsto E_1, \ldots, \mathbb{L}_{E_i} E_{i+1}, E_i, \ldots E_n.$$

  $\mathbb{L}$ is the twist functor.
- Given full exceptional collection $E_1, \ldots E_n$, we can extend it to the left by adding the object $\mathbb{L}_{E_1} \ldots \mathbb{L}_{E_{n-1}} E_n$. Then the n leftmost elements form a FEC, and we do the same. Extend in both directions to get an infinite collection of objects, called the helix of $E_1, \ldots E_n$.
- We have $E_{i-n} = S(E_i)$ (up to shift), where S is the Serre functor.
- Left (resp. right) dual collection is obtained by applying mutations to the right (resp. left) until the order is fully reversed.
- The initial A_∞-subcategory determines all the others.

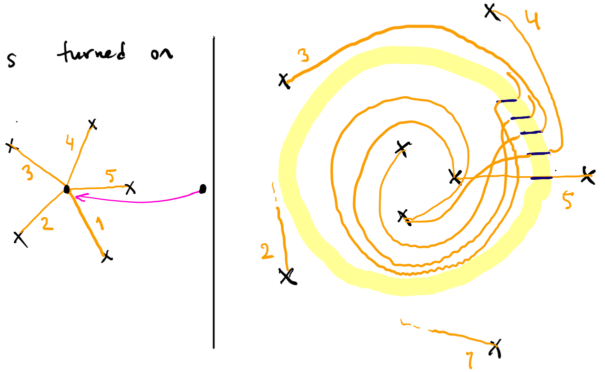
The main theorem

- $(\mathcal{A}_\emptyset, e_\emptyset) = (\mathcal{B}_\emptyset, e_\emptyset)$
- Theorem (Polishchuk - V): $(\mathcal{B}_a, e_a)$ can be obtained from $(\mathcal{B}_{a-}, e_{a-})$ by $\mathcal{R}$. Shifts can be computed.
- Theorem* (Polishchuk - V): $(\mathcal{A}_a, e_a)$ can be obtained from $(\mathcal{A}_{-a}, e_{-a})$ by $\mathcal{R}$. Only up to undetermined shifts.
- This proves Takahashi's HMS conjecture.
- Equivariant version in the interesting case $(d(a), \mu(a)) \neq 1$ needs a bit more work.

- Compute the vanishing cycles of the dual basis in

$$f_{s,r} : \{(z_1,\dots,z_n) \mid z_1 - sz_2 - rz_1^{a_1}z_2 - p_{-a}(z_2,\dots z_n) = 0\} \xrightarrow{z_1} \mathbb{C},$$

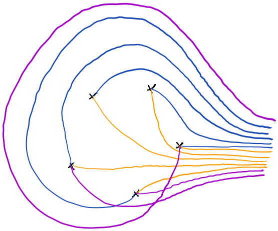
as matching cycles. s is a small positive real, $r = 1$ for now. Sign changes for convenience.

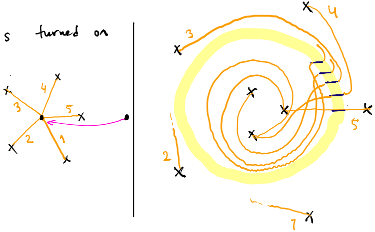


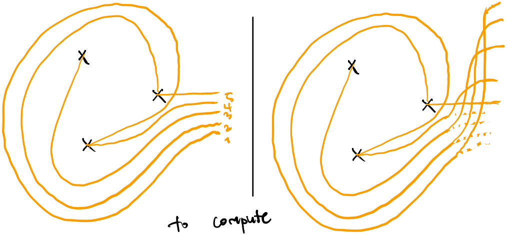
Recursion in the A-side II

- First get the vanishing cycle for the positive real critical value using real solutions.
- For each other critical value the vanishing cycle is similarly easy to compute for s with a specific argument. Then we move the s back to positive real axis by rotating it clockwise (and use braid parallel transport).
- The key point is that as s rotates, the inner critical values of $f_{s,1}$ are very close to rotating with it (with different speed) and the outer ones don't move much - rigorously proved using Rouche's theorem.
- Now fix the s, and let $r \to 0$. Inner critical values don't move much, outer ones go off the infinity roughly radially.
- When $r = 0$, the map is the same as $sz_2 + p_{-a}$ and we have part of the helix of the distinguished collection in maximally moved position (up to grading that we cannot yet compute).

Recursion in the A-side III







to compute
directed A_∞ subc.

Recursion in the B-side (only the first step)

- Consider $W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_n^{a_n}x_{n+1}^{a_n}$. We also define a $\mathbb{G}_m$-action on $\mathbb{C}^{n+1}$ leaving W invariant
- The VGIT machinery of Ballard-Favero-Katzarkov gives the diagram (except Φ)

$$\begin{tikzcd}
\mathcal{W}_{I^-} \arrow[r] \arrow[d, "r_-"', "\sim"] & \mathcal{W}_{I^+} \arrow[d, "r_+"', "\sim"] \\
\mathrm{MF}_{\Gamma_-}(w_-) \arrow[r, "\Phi"] & \mathrm{MF}_{\Gamma_+}(w_+)
\end{tikzcd}$$

where $w_- = W|_{x_n=1}$ and $w_+ = W|_{x_{n+1}=1}$, and the categories at the top row are "window subcategories" of $MF_\Gamma(W)$.

- We get a functor $MF_{\Gamma_{a-}}(p_{a-}) \to MF_{\Gamma_a}(p_a)$.

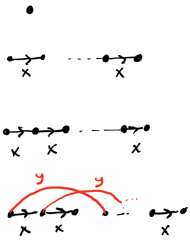
$a = \phi$:

$a = (a_1)$: $x^2 = 0$. $\deg x = 1$

$a = (a_1, a_2)$: $x^{a_2} = 0$ $\deg x = 0$

$a = (a_1, a_2, a_3)$: $x^2 = 0$ $\deg x = 1$ $y^{a_2} = 0$

$y = \mu_{a_1 - 1}(x, x, \ldots x)$



Conjecture: The endomorphism algebra of $\mathcal{A}_a$ is generated as a vector space by elements obtained by applying the A_∞-operations iteratively to x's. There are relations between such elements starting from $n = 4$.

References - with links

- Original Berglund-Hubsch paper (1993)
- Berglund-Henningson (1994)
- Takahashi, HMS (2007)
- Aramaki-Takahashi, B-side K_0 level (2019)
- Varolgunes, A-side K_0 level (2020)
- Ballard-Favero-Katzarkov, VGIT (2012)
- Seidel, Directed Fukaya-Seidel $Lag^{\rightarrow}(\mathbb{V})$ (2000)
- Seidel, $Fuk(p_a)$ by localization (2018)
- Gorodentsev-Kuleshov, Helix theory in triangulated categories (developed in late 80's)
- Seidel's book for mutations in A_∞-categories and matching cycles (no link)