Algebraic Geometry at Sea

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Solitons

The first description of a soliton was by the Scottish engineer John Scott Russell in 1834: he was conducting experiments on boats on a channel when he observed

“ *a mass of water [...] assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.* ”

Figure: A reconstruction of Russell’s observation in 1995 (Heriot-Watt University).

The KdV equation

Since this first sighting, solitary waves have been source of fascination among scientists, and they are still a very active topic of research.

They were described in mathematical terms by Boussinesq (1877) and Korteveg and de Vries (1895), as particular solutions of the Korteweg - de Vries equation:

$$4u_t - 6uu_x - u_{xxx} = 0$$

This is a famous nonlinear partial differential equation, which describes more generally the motion of waves in a one-dimensional shallow channel, such as the one of Russell.

A not so shallow channel

Figure: The Bosphorus strait this morning.

Translation waves

Let's look for translation waves solutions to the KdV equation: something of the form

$$u(x,t) = v(x - a \cdot t)$$

where v is a very nice function, for example analytic. Imposing that this is a solution we find

$$-4a \cdot v' - 6v \cdot v' - v''' = 0.$$

Integrating we get

$$-4a \cdot v - 3(v)^2 - v'' + b = 0.$$

We can multiply by v' to obtain

$$-4a \cdot vv' - 3(v)^2 v' - v''v' + b \cdot v' = 0,$$

which we can integrate again to obtain

$$-2a \cdot v^2 - v^3 - \frac{1}{2}(v')^2 + b \cdot v + c = 0$$

The Weierstrass equation

To summarize, we need to solve the differential equation

$$(v')^2 = -4v^3 - 4a \cdot v^2 + 2b \cdot v + 2c$$

Up to a constant, this is the same as the famous Weierstrass equation:

$$y^2 = f(x), \qquad f(x) = -4x^3 - 4a \cdot x^2 + 2b \cdot x + 2c$$

Geometrically this cuts out an elliptic curve $E = \{(x, y) \mid y^2 = f(x)\}$ in $\mathbb{C}^2$ and we are looking for an analytic function $v \colon \mathbb{C} \to \mathbb{C}$ such that we have a well-defined map

$$\mathbb{C} \longrightarrow E \qquad z \mapsto (v(z), v'(z))$$

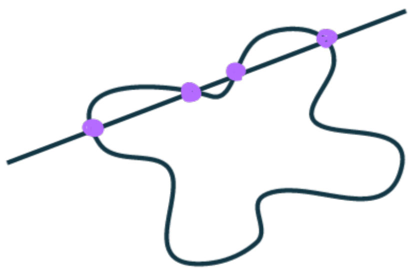
But such a function is known since more than 100 years: it is the Weierstrass $\wp$-function, or, equivalently, the (second logarithmic derivative of the) theta function of the curve E.

This discussion works much more generally.

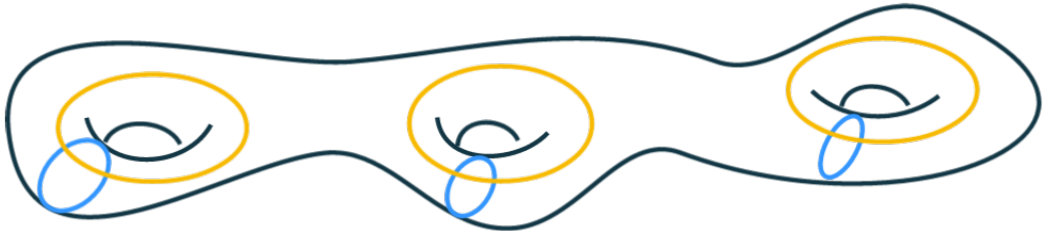
Algebraic curves and Riemann surfaces

An algebraic curve is a variety of dimension one, the zero locus of polynomial equations in $\mathbb{C}^n$ or $\mathbb{P}^n$.

$$\{x^4 + y^4 = 1\}$$



Smooth curves correspond to Riemann surfaces: complex manifolds of dimension one.



The KP equation

Algebraic curves are connected to the Kadomtsev-Petviashvili (KP) equation:

$$\frac{\partial}{\partial x}(4u_t - 6uu_x - u_{xxx}) = 3u_{yy}$$

which describes the evolution of two-dimensional waves in shallow water.

This is a generalization of the KdV equation

$$4u_t - 6uu_x - u_{xxx} = 0$$

Figure: KP equation in French waters (Wikipedia).

Riemann's theta function

The connection goes through Riemann's theta function. Integrating holomorphic differentials on a genus g curve, we obtain the Riemann matrix:

$$\tau \text{ ``}=\text{''} \left(\int_{b_j} \omega_i \right)$$

which is a complex symmetric $g \times g$ matrix with positive definite imaginary part.

Riemann's theta function The associated theta function is a function of $z \in \mathbb{C}^g$ given by

$$\theta(z,\tau) = \sum_{n \in \mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2} n^t \tau n\right) \cdot \mathbf{e}(n^t z) = \sum_{n \in \mathbb{Z}^g} a_n(\tau) \cdot \mathbf{e}(n^t z).$$

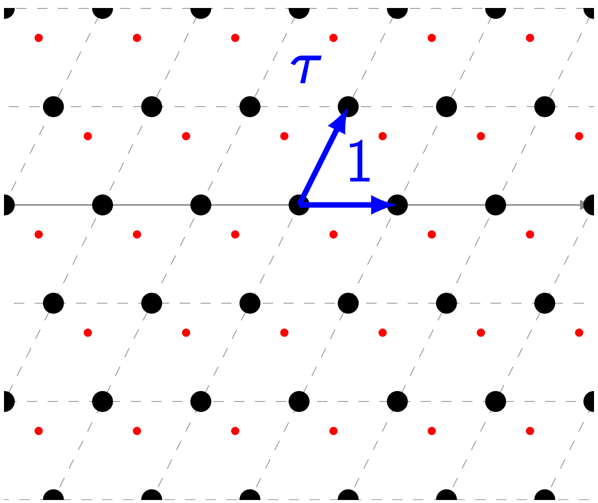
where $\mathbf{e}(x) = \exp(2\pi i x)$. This is an infinite sum of exponentials, and it is computable: Maple, MATLAB, Sage, Julia,...

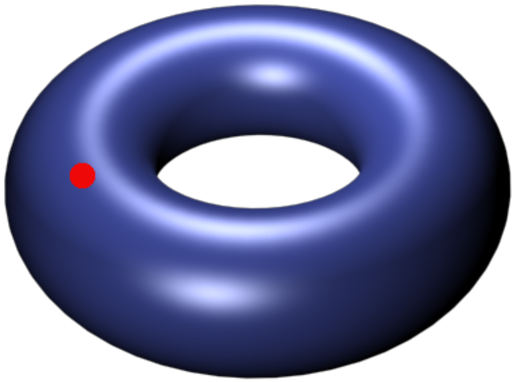
Abelian varieties and Jacobians

In geometric terms, the Riemann matrix τ corresponds to an abelian variety:

$$A_\tau = \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$$

We can construct an abelian variety starting from any Riemann matrix τ. If this comes from a curve, the abelian variety is called the Jacobian of the curve C.





Curves and waves

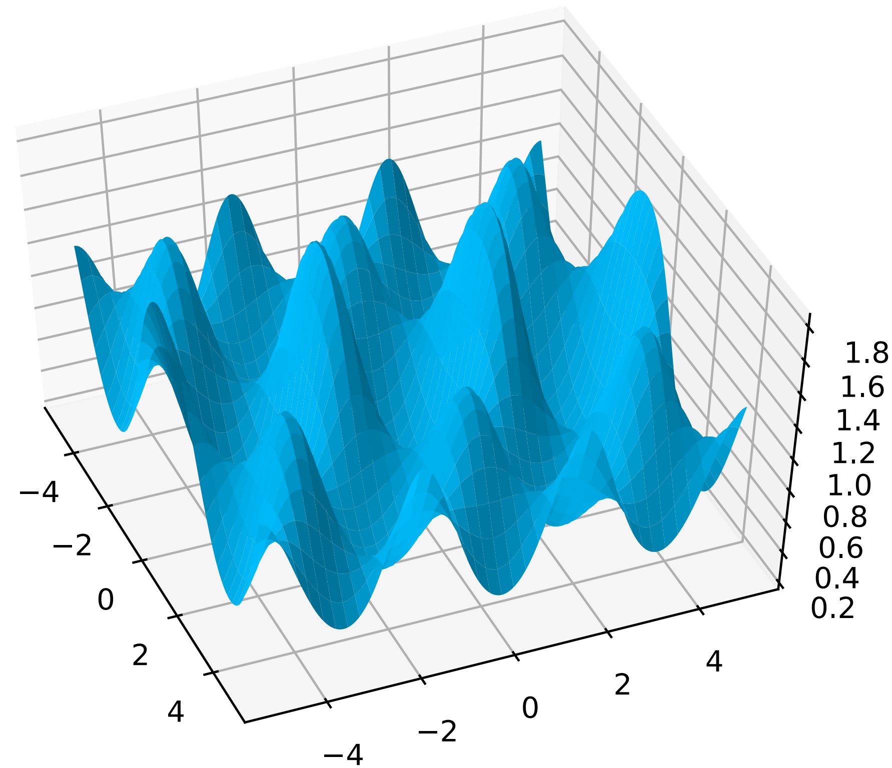
Krichever proved that the KP equation has a solution of the form

$$u(x,y,t) = 2\frac{\partial^2 \log}{\partial^2 x}\theta(\mathbf{U}x + \mathbf{V}y + \mathbf{W}t, \tau)$$

for certain vectors $\mathbf{U}, \mathbf{V}, \mathbf{W} \in \mathbb{C}^g$.

Shiota then solved the Schottky problem: a matrix τ represents a Jacobian if and only if the above is a solution of the KP equation.

The first result is very concrete: on the right, a wave obtained from a quartic plane curve in Julia.



Many research directions

- Classify the KP solution arising from a fixed curve: the Dubrovin Threefold. Together with T. Çelik and B. Sturmfels.
- What happens when the curve degenerates to a singular nodal curve? Tropical geometry. Together with C. Fevola, Y. Mandelshtam and B. Sturmfels.
- What happens when the curve degenerates to a cuspidal curve? Rational solutions. Together with T. Çelik and J. Little.
- Try to recover the curve from the matrix τ: Effective Torelli theorem. Together with T. Çelik and D. Eken.

Singular curves

One proof of Krichever's theorem is based on two essential ingredients:

1. The algebraic structure of the KP equation: **Sato Grassmannian** of Sato, Segal-Wilson.
2. **Abel's theorem** and Riemann-Roch.

In particular, it can be replicated on singular curves as well: since Abel's theorem and Riemann- Roch still hold.

Example: soliton solutions

Consider the nodal cubic $C = \{(x, y) \mid y^2 = x^3 + x^2\} \subseteq \mathbb{C}^2$. This is the image of

$$\mathbb{C} \longrightarrow C, \qquad t \mapsto \left(t^2 - 1, t^3 - t\right)$$

A basis of canonical differentials is given by $\omega = \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt$, and $\int_\gamma \omega = 2\pi i$. Hence

$$\mathrm{Jac}(C) = \mathbb{C}/2\pi i\mathbb{Z} \xrightarrow{\sim} \mathbb{C}^*, \quad z \mapsto \exp(z)$$

The theta function in this case is

$$\theta(z) = \exp(z) - 1$$

This is a **finite** sum of exponentials, and gives rise to a **soliton solution**.

Note that the nodal cubic $y^2 = x^3 + x^2$ is a special, or degenerate, case of the cubic $y^2 = f(x)$.

Singular solutions and tropical geometry

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Fevola, Mandelstham, Sturmfels and myself, studied these degenerations and the corresponding solitons, from the point of view of tropical geometry.

Theorem (A., Fevola, Mandelstham, Sturmfels - 2021; Ichikawa - 2023) Under the degeneration to a rational nodal curve, the theta function becomes a finite sum of exponentials:

$$\theta(z) = \sum_{n\in\mathbb{Z}^g} \exp\left(\pi i n^T \tau n + 2\pi i n^T z\right) \rightsquigarrow \hat{\theta} = \sum_{c\in\mathcal{C}} a_c \exp(c^T z)$$

The corresponding solution of the KP equation is a soliton solution.

Example: rational solutions

Consider the cuspidal cubic $C = \{(x, y) \mid y^2 = x^3\} \subseteq \mathbb{C}^2$. This is the image of

$$\mathbb{C} \longrightarrow C, \qquad t \mapsto \left(t^2, t^3\right)$$

A basis of canonical differentials is given by $\omega = \frac{1}{t^2} dt$, and $\int_\gamma \omega = 0$. Hence

$$\mathrm{Jac}(C) = \mathbb{C}$$

The theta function in this case is

$$\theta(z) = z.$$

This is a polynomial, and gives rise to a **rational solution**.

Note that the cuspidal cubic $y^2 = x^3$ is an even more degenerate, case of the cubic $y^2 = f(x)$.

Polynomial theta functions

Together with Çelik and Little, we have classified which curves give rise to polynomial theta functions.

Theorem (A., Çelik, Little - 2021) The theta function of C is polynomial if and only if the curve is rational with unibranched singularities. Moreover, the resulting polynomial is of degree at most $\frac{g(g+1)}{2}$ and gives a rational solution to the KP equation.

In general we have:

Curve	Theta function	KP solution
smooth	infinite sum of exponentials	quasiperiodic
nodal rational	finite sum of exponentials	soliton
unibranched rational	polynomial	rational

More research directions

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- When do we get real soliton solutions out of rational nodal curves? Tropical geometry and positive geometry. Work in progress by S. Abenda, T. Çelik, C. Fevola, Y. Mandelshtam.
- When do we get real rational solutions out of rational unibranched curves? Discussion in progress with T. Çelik and J. Little.
- What about real real solutions?

What about "actual" applications?

What if we want to actually model water waves this way? A natural idea:

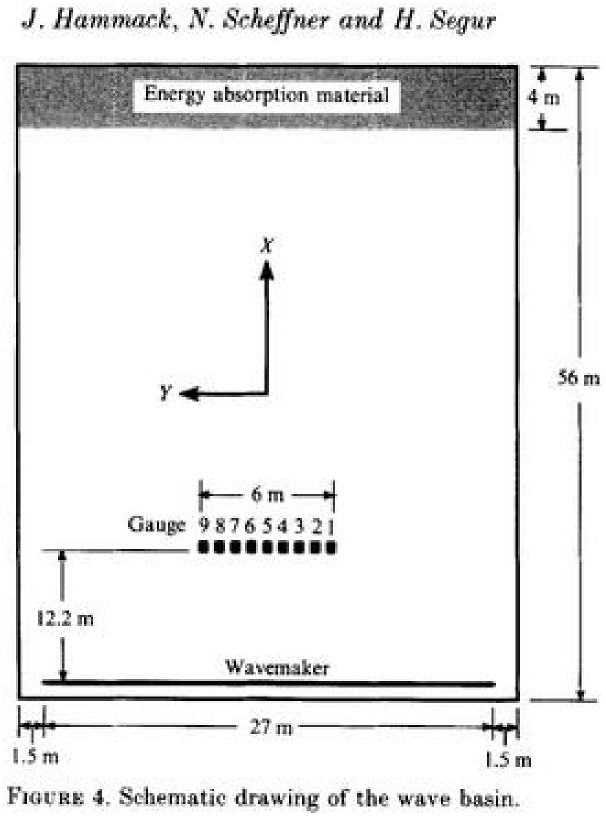

Figure: from Hammack, Scheffner, Segur, Journal of Fluid Mechanics (1989)

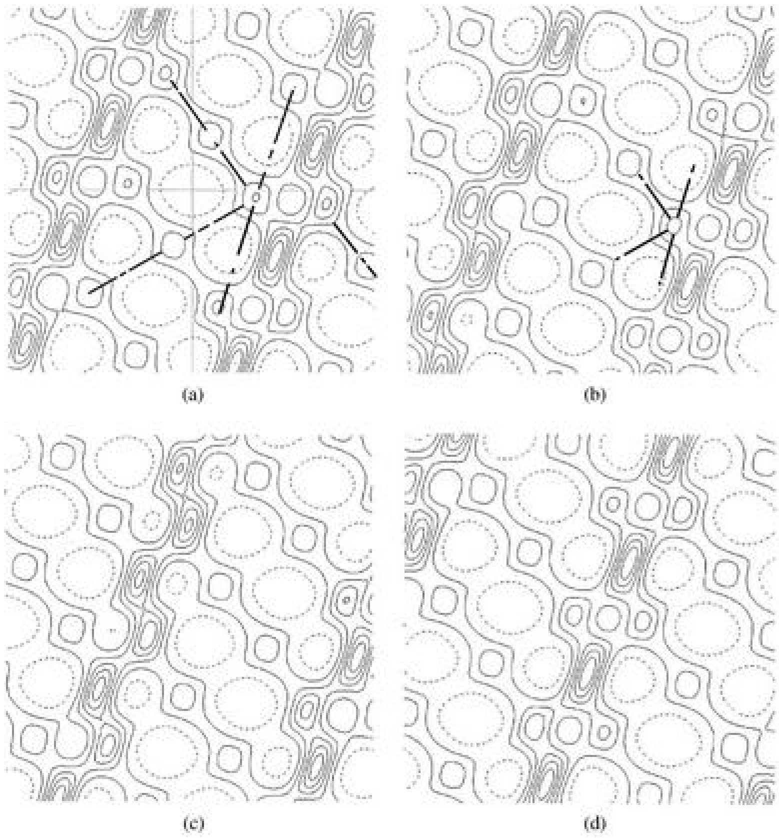

Figure: from Dubrovin, Flickinger, Segur, Studies in applied mathematics (1997)

And today?

In a joint work in progress with M. Bennett, T. Çelik, B. Deconinck and C. Wang, we consider the problem of reconstructing the data of a Riemann matrix τ, together with the wave vectors $\mathbf{U}, \mathbf{V}, \mathbf{W}$, from one KP solution

$$u(x, y, t) = 2\frac{\partial^2 \log}{\partial^2 x}\theta(\mathbf{U}x + \mathbf{V}y + \mathbf{W}t, \tau)$$

This was already done by Dubrovin, Flickinger and Segur, but only with genus two solutions, and with techniques valid exclusively in genus two.

An algorithm

- Suppose f is a periodic function of x with period U, then we have

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \cdot \exp\left(2\pi i \cdot k \cdot Ux\right), \qquad \hat{f}(\xi) = \sum_{k\in\mathbb{Z}} c_k \delta(\xi - 2\pi U k).$$

  So, if the coefficients c_k decay sufficiently fast, the Fourier transform of f has "peaks" around $2\pi k$, for small values of k.
- We look the Fourier transform of $u(x, 0, 0)$. This will have some "peaks", which turn out to be the coordinates of $\mathbf{U}$. Same with y, z and $\mathbf{V}, \mathbf{W}$ respectively.
- We can recover the function $\tau(x, y, z) = \theta(\mathbf{U}x + \mathbf{V}y + \mathbf{W}z, B)$ from U.
- Recovering the matrix τ becomes a matter of numerical linear algebra:

$$\tau(x, 0, 0) = \sum_{n\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2} n^t \tau n\right) \cdot \mathbf{e}(n^t \mathbf{U} x)$$

Thank you!