Period integrals of hypersurfaces via tropical geometry

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Background

Gamma conjecture for Calabi–Yau manifolds

- (X, ω): a Calabi–Yau manifold with a symplectic structure
- $\{Z_t\}_{t \in \Delta^*}$: a family of Calabi–Yau manifolds mirror to (X, ω)
- $C_t \subset Z_t$: a Lagrangian cycle
- E: a coherent sheaf on X mirror to C_t

Then one has

$$\int_{C_t} \Omega_t = \int_X t^{-\omega} \cdot \widehat{\Gamma}_X \cdot \left(2\pi\sqrt{-1}\right)^{\frac{\deg}{2}} \cdot \mathrm{ch}(E) + O\left(t^{\epsilon}\right) \quad (t \to +0),$$

where

- Ω_t: the holomorphic volume form on Z_t, and
- $\widehat{\Gamma}_X \in H^*(X, \mathbb{R})$: the gamma class of X.

Gamma classes

- The *gamma class* of X is defined by

$$\widehat{\Gamma}_X := \prod_i \Gamma\left(1+\delta_i\right) \in H^*\left(X, \mathbb{R}\right),$$

where δ_i are the Chern roots of X, i.e., $c(TX) = \prod_i (1+\delta_i)$.

- One can write it as

$$\widehat{\Gamma}_X = \exp\left(-\gamma c_1(TX) + \sum_{k=2}^{\infty} (-1)^k \left(k-1\right)! \zeta(k) \operatorname{ch}_k(TX)\right),$$

where

  - $\gamma = \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n} 1/k\right)$: the Euler's constant, and
  - $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$: the Riemann zeta value.

Gamma conjecture and tropical geometry

Theorem (Abouzaid–Ganatra–Iritani–Sheridan=:AGIS)

One can compute the asymptotics of $\int_{C_t} \Omega_t$ $(t \to +0)$ in the case where

- X, Z_t: a mirror pair of Calabi–Yau hypersurfaces of Batyrev
- C_t: a cycle mirror to an ambient line bundle E on X

by using *tropical geometry*, and can see that

$$\int_{C_t} \Omega_t = \int_X t^{-\omega} \cdot \widehat{\Gamma}_X \cdot \left(2\pi\sqrt{-1}\right)^{\frac{\deg}{2}} \cdot \mathrm{ch}(E) + O\left(t^{\epsilon}\right) \quad (t \to +0)$$

holds in this case.

Goal/Plan of the talk

Goal

To generalize the computation by AGIS to the case where

- the hypersurfaces are not necessarily Calabi–Yau hypersurfaces (geometric genus > 1), and
- the integrands are the Poincaré residues of meromorphic forms having poles of higher order along the hypersurfaces.

Plan

- Tropical geometry
- Idea of computations by AGIS
- Sketch of its generalization
- Main result

Overview of tropical geometry

- Tropical geometry is algebraic geometry over the *tropical number semi-field* $(\mathbb{T} := \mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$$a \oplus b := \min\{a, b\}$$
$$a \odot b := a + b.$$

- We study *tropical varieties* which are tropical counterparts of algebraic varieties.
- Tropical varieties are polyhedral complexes equipped with a certain kind of integral affine structures.
- Tropical varieties arise as limits of degenerating families of complex algebraic varieties.

Tropical hypersurfaces

- (K, val): a valued field
- For a Laurent polynomial

$$f = \sum_{m=(m_1,\cdots,m_{d+1})} k_m z_1^{m_1} \cdots z_{d+1}^{m_{d+1}} \in K\left[z_1^{\pm}, \cdots, z_{d+1}^{\pm}\right],$$

the following are defined:

Definition

- The *tropicalization* $\mathrm{trop}(f)$ of f is the piecewise affine function $\mathbb{R}^{d+1} \to \mathbb{R}$ defined by

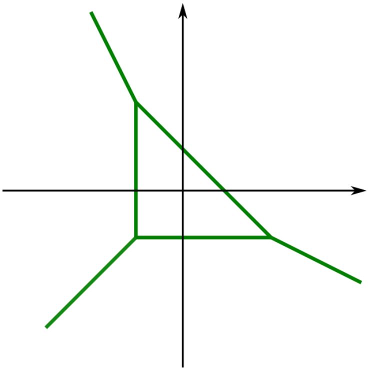
$$\mathrm{trop}(f)(Z_1, \cdots, Z_{d+1}) := \min_m \{\mathrm{val}(k_m) + m_1 Z_1 + \cdots + m_{d+1} Z_{d+1}\}.$$

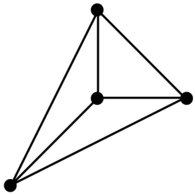
- The *tropical hypersurface* $V(\mathrm{trop}(f)) \subset \mathbb{R}^{d+1}$ defined by $\mathrm{trop}(f)$ is the corner locus of $\mathrm{trop}(f)$.

Example

- $K := \mathbb{C}\{x\}$: the convergent Laurent series field
- $\mathrm{val}\colon K \longrightarrow \mathbb{Z} \cup \{-\infty\}, \quad \sum_{j\in\mathbb{Z}} c_j x^j \mapsto \min\{j \in \mathbb{Z} \mid c_j \neq 0\}$
- $f := 1 + x\left(z_1 + z_2 + z_1^{-1} z_2^{-1}\right) \in K\left[z_1^{\pm}, z_2^{\pm}\right]$

$$\rightsquigarrow \mathrm{trop}(f)(Z) = \min\{0, 1 + Z_1, 1 + Z_2, 1 - Z_1 - Z_2\}$$





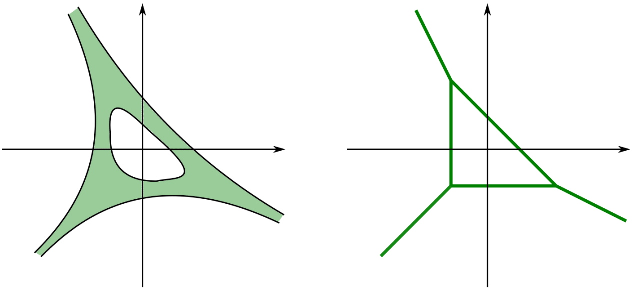
Amoebas of complex hypersurfaces

- $0 < t << 1 \rightsquigarrow f_t := f|_{x=t} \in \mathbb{C}[z_1^{\pm}, \cdots, z_{d+1}^{\pm}]$
- $Z_t := \{z \in (\mathbb{C}^*)^{d+1} \mid f_t(z) = 0\}$
- $\mathrm{Log}_t \colon (\mathbb{C}^*)^{d+1} \to \mathbb{R}^{d+1}$

$$(z_1, \ldots, z_{d+1}) \mapsto (\log_t |z_1|, \ldots, \log_t |z_{d+1}|)$$

Theorem (Mikhalkin, Rullgård)

$$\lim_{t\to 0} \mathrm{Log}_t(Z_t) = V(\mathrm{trop}(f)).$$



Sketch of the proof

Recall that for a Laurent polynomial $f = \sum_m k_m z^m$,

$$\mathrm{trop}(f)(Z_1, \cdots, Z_{d+1}) := \min_m \{\mathrm{val}(k_m) + m \cdot Z\}.$$

Take a point $Z \in \mathbb{R}^{d+1}$. On $\mathrm{Log}_t^{-1}(Z)$,

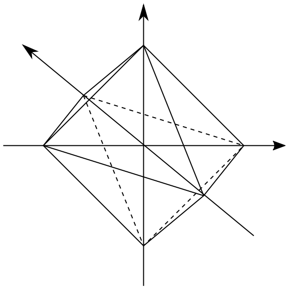
- we have $k_m z^m|_{x=t} = O\left(t^{\mathrm{val}(k_m)+m\cdot Z}\right)$, and
- monomials $k_m z^m|_{x=t}$ such that

$$\mathrm{val}(k_m) + m \cdot Z = \mathrm{trop}(f)(Z)$$

  are leading terms of f_t.

Sketch of computations by AGIS

Example



- $M := \bigoplus_{i=1}^{3} \mathbb{Z} e_i$
- $f_t(z) := -1 + \sum_{m \in (\Delta \cap M) \setminus \{0\}} t \cdot z^m$
- $Z_t := \left\{ z \in (\mathbb{C}^*)^3 \,\middle|\, f_t(z) = 0 \right\}$
- $C_t := Z_t \cap (\mathbb{R}_{>0})^3$
- $\Omega_t := \frac{1}{df_t} \bigwedge_{i=1}^{3} \frac{dz_i}{z_i}$
- $V(\mathrm{trop}(f)) \subset \mathbb{R}^3$

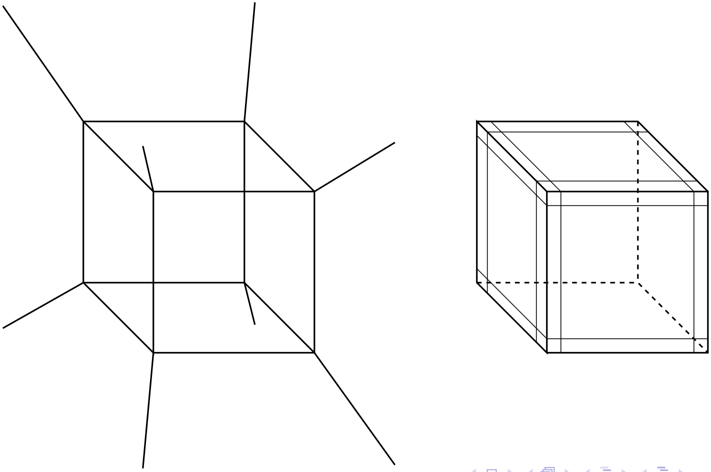
We consider

- $i_t : \mathbb{R}^3 \to (\mathbb{R}_{>0})^3, \quad (Z_1, Z_2, Z_3) \mapsto \left(t^{Z_1}, t^{Z_2}, t^{Z_3}\right)$
- $B_t := i_t^{-1}(C_t)$

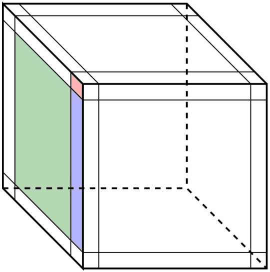
and try to compute $\int_{C_t} \Omega_t = \int_{B_t} i_t^* \Omega_t$.

Sketch of computations by AGIS (continued)

- $B_t := i_t^{-1}(C_t)$ converges to the central part of $V(\mathrm{trop}(f))$.
- Decompose B_t into pieces according to the polyhedral structure of $V(\mathrm{trop}(f))$.



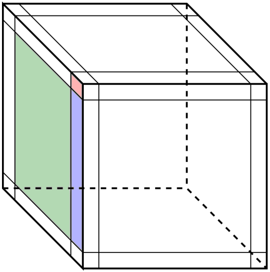
Sketch of computations by AGIS (continued)



We can simplify $f_t(z) = -1 + \sum_{m \in (\Delta \cap M) \setminus \{0\}} t \cdot z^m = 0$ to

- $1 = tz_1 \left(1 + O\left(t^\epsilon\right)\right)$ on i_t (green),
- $1 = tz_1 \left(1 + tz_2/tz_1 + O\left(t^\epsilon\right)\right)$ on i_t (blue), and
- $1 = tz_1 \left(1 + tz_2/tz_1 + tz_3/tz_1 + O\left(t^\epsilon\right)\right)$ on i_t (red).

Sketch of computations by AGIS (continued)



Using the simplified equation on each region, we get

- $\int_{\color{green}\blacksquare} i_t^* \Omega_t = \mathrm{vol}\left({\color{green}\blacksquare}\right) \cdot (-\log t)^2 + O\left(t^\epsilon\right)$,
- $\int_{\color{blue}\blacksquare} i_t^* \Omega_t = \mathrm{vol}\left({\color{blue}\blacksquare}\right) \cdot (-\log t)^2 - \zeta(2) + O\left(t^\epsilon\right)$, and
- $\int_{\color{red}\blacksquare} i_t^* \Omega_t = \mathrm{vol}\left({\color{red}\blacksquare}\right) \cdot (-\log t)^2 + O\left(t^\epsilon\right)$.

Sketch of computations by AGIS (continued)

In total, we obtain

$$\int_{B_t} i_t^* \Omega_t = \mathsf{vol}\,(\text{the central sphere}) \cdot (-\log t)^2 - 24 \cdot \zeta(2) + O\left(t^{\epsilon}\right)$$
$$= \int_X t^{-\omega} \cdot \widehat{\Gamma}_X + O\left(t^{\epsilon}\right),$$

where

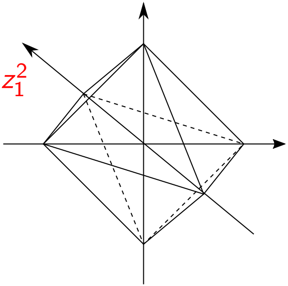
- (X, ω): an anticanonical hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with the anticanonical polarization, and
- $(E = \mathcal{O}_X)$.

Generalization of the computations

Theorem (Y., rough version)

Similar computations are possible also for the case where the Newton polytope Δ has more lattice points in its interior.

Example



- $f_t := -1 + \sum_{m \in (\Delta \cap M) \setminus \{0\}} t \cdot z^m + t^3 z_1^2$
- $C_t := Z_t \cap (\mathbb{R}_{>0})^3$
- $\Omega_t := t z_1 \cdot \frac{1}{df_t} \bigwedge_{i=1}^{3} \frac{dz_i}{z_i}$
- $V(\mathrm{trop}(f)) \subset \mathbb{R}^3$

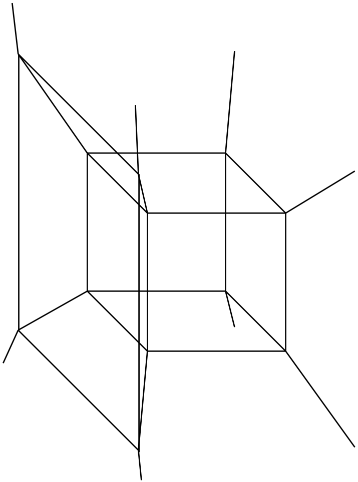
We consider

- $i_t \colon \mathbb{R}^3 \to (\mathbb{R}_{>0})^3, \quad (Z_1, Z_2, Z_3) \mapsto \left(t^{Z_1}, t^{Z_2}, t^{Z_3}\right)$
- $B_t := i_t^{-1}(C_t)$

and try to compute $\int_{C_t} \Omega_t = \int_{B_t} i_t^* \Omega_t$.

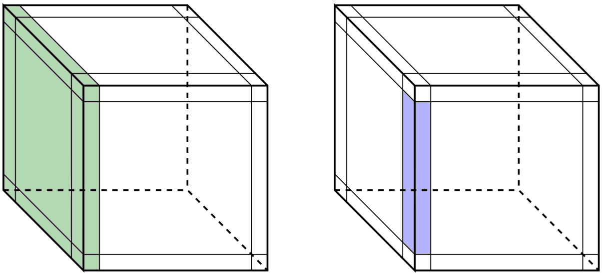
Generalization of the computations (continued)

- $V\left(\text{trop}\left(f\right)\right) \subset \mathbb{R}^3$



- $B_t := i_t^{-1}\left(C_t\right)$ converges to the boundary of the right cube in $V\left(\text{trop}\left(f\right)\right)$ again.

Generalization of the computations (continued)



Highlight

- Integrals only over the above green regions are effective.
- Around an edge, we get

$$\int_{\blacksquare+\blacksquare} i_t^*\Omega_t = \mathrm{vol}\left(\blacksquare\right)\cdot(-\log t)^2 - \zeta(2) + O\left(t^{\epsilon}\right).$$

Generalization of the computations (continued)

In total, we obtain

$$\int_{B_t} i_t^*\Omega_t = \mathrm{vol}\,(\text{facet}) \cdot (-\log t)^2 - 4 \cdot \zeta(2) + O\left(t^{\epsilon}\right)$$
$$= \int_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1} t^{-\omega} \cdot D_1 \cdot \widehat{\Gamma}_0 + O\left(t^{\epsilon}\right),$$

where

- $D_1 := \{0\} \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and
- $\widehat{\Gamma}_0 := \frac{\prod_i \Gamma(1+D_i)}{\Gamma(1+\sum_i D_i)}$ $\quad$ ($\{D_i\}_i$: all toric divisors on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$).

Remark

The restriction of $\widehat{\Gamma}_0$ to an anticanonical hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ coincides with the gamma class of the hypersurface.

Setup of the main result

- $\Delta \subset M_{\mathbb{R}}$: a lattice polytope of dimension $d+1$ $(d \geq 1)$
- $\Sigma \subset N_{\mathbb{R}}$: a simplicial refinement of the normal fan of Δ
- Y_{Σ}: the toric variety over $\mathbb{C}$ associated with Σ
- $w \in \operatorname{Int}(\Delta) \cap M$
- Consider the hypersurface $Z_t \subset Y_{\Sigma}$ defined by the polynomial

$$f_t(z) = -t^{\lambda_w} z^w + \sum_{m \in (\Delta \cap M) \setminus \{w\}} t^{\lambda_m} z^m \quad (\lambda_m \in \mathbb{Z}).$$

  We suppose $(\Delta \cap M) \to \mathbb{Z}, m \mapsto \lambda_m$ extends to a strictly convex affine function on a unimodular triangulation $\mathscr{T}$ of Δ.
- $C_t := Z_t \cap (\mathbb{R}_{>0})^{d+1}$

Setup of the main result (continued)

- $l \cdot \Delta := \{l \cdot m \in M_{\mathbb{R}} \mid m \in \Delta\}$ $(l \in \mathbb{Z}_{>0})$
- $v \in \operatorname{Int}(l \cdot \Delta) \cap M$
- $\tau_v \in \mathscr{T}$: the minimal cell such that $v \in l \cdot \tau_v$
  $\rightsquigarrow$ $v = \sum_{m \in \tau_v \cap M} p_m \cdot m \quad (p_m \in \mathbb{Z}_{>0}, \sum_m p_m = l)$
- $\omega_t^{l,v}$: the meromorphic $(d+1)$-form on Y_Σ defined by

$$\omega_t^{l,v} := (l-1)! \left(\bigwedge_{i=0}^{d} \frac{dz_i}{z_i} \right) \frac{z^v}{(f_t)^l} \prod_{m \in \tau_v \cap M} t^{p_m \lambda_m}$$

  The forms $\left\{\omega_t^{l,v}\right\}_v$ generate $H^0\left(Y_\Sigma, \Omega^{d+1}(l \cdot Z_t)\right)$.
- $\Omega_t^{l,v} \in H^d(Z_t, \mathbb{C})$: the Poincaré residue of $\omega_t^{l,v}$, i.e., the image by the Poincaré residue map

$$\operatorname{Res}\colon H^0\left(Y_\Sigma, \Omega^{d+1}(l \cdot Z_t)\right) \to H^d(Z_t, \mathbb{C})$$

Main result

Theorem (Y., simplified version)

One has

$$\int_{C_t} \Omega_t^{l,v} = \begin{cases} \int_{Y_w} t^{-\omega_\lambda^w} \cdot E_{v,w} \cdot \widehat{\Gamma}_w + O\left(t^\epsilon\right) & \mathrm{conv}\left(\{w\} \cup \tau_v\right) \in \mathscr{T} \\ O\left(t^\epsilon\right) & \text{otherwise} \end{cases}$$

as $t \to +0$, for some $\epsilon > 0$, where

- Y_w: the toric variety associated with the fan

$$\Sigma_w := \{\mathbb{R}_{\geq 0} \cdot (\tau - w) \mid \tau \in \mathscr{T}, \tau \ni w\},$$

- $\omega_\lambda^w := \sum_{m \in A_w} (\lambda_m - \lambda_w) D_m^w$ with
  - $A_w := \{m \in (\Delta \cap M) \setminus \{w\} \mid \mathrm{conv}(\{m, w\}) \in \mathscr{T}\}$
  - D_m^w $(m \in A_w)$: the toric divisor on Y_w associated with the 1-dimensional cone $\mathbb{R}_{\geq 0} \cdot (m - w) \in \Sigma_w$,
- $E_{v,w} := \prod_{m \in A_w \cap \tau_v} \prod_{i=0}^{p_m - 1} \left(D_m^w + i\right) \cdot \prod_{i=0}^{p_w - 1} \left(\sum_{m \in A_w} D_m^w - i\right)$
- $\widehat{\Gamma}_w := \frac{\prod_{m \in A_w} \Gamma(1 + D_m^w)}{\Gamma\left(1 + \sum_{m \in A_w} D_m^w\right)}$.