SOME NOTATION

- (M,ω) closed symplectic manifold s.t.

$c_1(M)=0$ + trivialization of $\Lambda^{top}_{\mathbb{C}} TM$ / htpy.

- $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$ (Novikov field)
- $Q := \left(QH^*(M;\Lambda), *, (\,,\,) \right)$

- quantum cohomology Frobenius algebra.

GOAL

- Try to understand Q by a local-to-global method.
- Motivation: SYZ mirror symmetry.
- The same approach exists for big quantum cohomology & Fukaya category
- Could work over $\Lambda_{\geq 0}$ for finer statements

RELATIVE SYMPLECTIC COHOMOLOGY

- $K \subset M$ compact $\rightsquigarrow$ $SH^*_M(K;\Lambda)$
- graded unital commutative algebra.
- canonical presheaf structure
- $SH^*_M(\emptyset;\Lambda) = \{0\}$ & $SH^*_M(M;\Lambda) \cong Q$.
- <u>Def</u>: Take the cohomology of

$$SC^*_M(K;\Lambda) := \widehat{\underset{\substack{\longrightarrow \\ H|_K < 0,}}{\mathrm{hocolim}}} \; CF^*(H;\Lambda)$$

monotone Floer data

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INVOLUTIVE COVERS I

Def: $M = K_1 \cup \dots \cup K_N$ is called involutive if there exists $\{ f_i^m : M \xrightarrow{sm.} \mathbb{R}_{\geq 0} \}^{m=1,\dots,N}_{i=1,2,3,\dots}$ such that for every $m = 1, \dots, N$:

- $f_i^m |_{K_m} = 0$
- $f_1^m \leq f_2^m \leq \dots$ with $\bigcap_{i=1}^{\infty} \{ f_i^m = 0 \} = K_m$

and for every $i \in \mathbb{N}$, $m, m' \in [N]$:

- $\{ f_i^m , f_i^{m'} \} = 0$.

INVOLUTIVE COVERS II

Example: $\pi : M \to B$ Lagrangian torus fibration with singularities, $B = P_1 \cup \dots \cup P_N$, $K_m := \pi^{-1}(P_m)$

Example (better) $\mathcal{X} \to \mathbb{D}$ polarized semi-stable degeneration of compact complex varieties. Then, we have $\mathrm{spec} : X_1 \xrightarrow{\text{cts.}} X_0$. We take $M = X_1$ & $K_m = \mathrm{spec}^{-1}(X_0^m)$ for $X_0 = \bigcup_{m=1}^{N} X_0^m$.

(Mclean-Tehrani-Zinger, can allow some sing. in $X_0 \subset \mathcal{X}$)

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Example: $\pi : M \to B$ Lagrangian torus fibration (fibers Maslov 0) $B = P_1 \cup \dots \cup P_N$, $K_m := \pi^{-1}(P_m)$

Example (better) $X \to \mathbb{D}$ polarized semi-stable degeneration (with (X, X_0) log CY). Then, we have $\mathrm{spec} : X_1 \xrightarrow{\text{cts.}} X_0$. We take $M = X_1$ & $K_m = \mathrm{spec}^{-1}(X_0^m)$ for $X_0 = \bigcup_{m=1}^{N} X_0^m$.

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DESCENT I

- Let $M = \bigcup_{m=1}^{N} K_m$ be an involutive cover.

  Define $K_I := \bigcap_{m \in I} K_m$, for $I \subset [M]$

- Thm (V.): The canonical map

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$$\text{RHS} := \mathrm{Tot}\left(\begin{tikzcd} \uparrow & \uparrow & \\ \bigoplus_m SC^1_M(K_m) \arrow[r] \arrow[u] & \bigoplus_{m<m'} SC^1(K_{mm'}) \arrow[r] \arrow[u] & \cdots \\ \bigoplus_m SC^0_M(K_m) \arrow[r, "d_{Cech}"] \arrow[u] & \bigoplus_{m<m'} SC^0(K_{mm'}) \arrow[r] \arrow[u] & \\ \arrow[u, "d_{Floer}"'] & \arrow[u] & \end{tikzcd}\right)$$

DESCENT II

- <u>Thm</u> (Abouzaid - Groman - V.) : $SC^*_M(K;\Lambda)$ can be equipped with a natural BV_∞-structure.

- <u>Cor</u> : The Cech complex can be equipped with a BV_∞-structure and the local-to-global map is a BV_∞-map.

- In particular, the homology level local-to-global isomorphism respects algebra structures.

THE HEADACHE

- All the operations that can be defined on $SH^*_M(K;\Lambda)$ for $K \neq M$ must have at least one output (no a priori lower bound on topological energy, does not descend to completion)

(possibly ∞-dim)

- There is no local trace / pairing operation; only the global one on $Q \cong SH_M(M;\Lambda)$.

MIRROR SYMMETRY I

Claim: For certain involutive covers $M = \bigcup_{m=1}^{N} K_m$ (including Gross-Siebert toric degenerations) we can construct smooth proper rigid analytic space $\mathcal{Y}$ with an affinoid cover $\bigcup_{m=1}^{N} Y_m$ & analytic vol. form Ω s.t. for $0 \neq I \subset [M]$

$$SC^*_M(K_I; Y) \xrightarrow[BV_\Omega]{\cong} T(Y_I; \Lambda^* T\mathcal{Y})$$

compatibly with restriction maps

cts poly vector fields on $M(\mathcal{O}(Y_i))$.

(see Groman-V. for open M)

MIRROR SYMMETRY II

Cor: We obtain an isomorphism of algebras

$$Q \cong \left(H_{dR}(Y), \; \iota_\Omega \circ \wedge \circ \iota_\Omega^{-1} \right)$$

Rmk: • $SH^0_M(K_I; \Lambda) \cong \mathcal{O}(Y_I)$ by construction.

For full statement, we rely on the existence of a locally generating "homological section" $L \subset M$ and generalization of DTT formality.

• Poincaré trace $\xleftrightarrow{\text{mirror}}$ Grothendieck trace. ?

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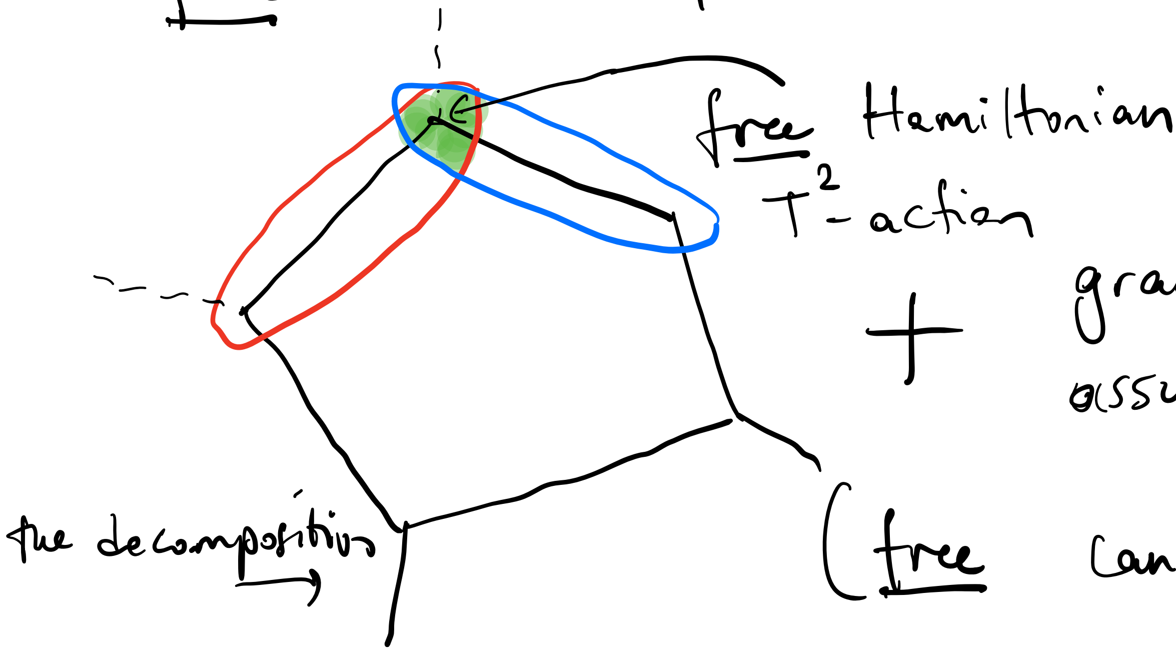
NEEDS WORK

• Poincare trace $\xleftrightarrow{\text{mirror}}$ Grothendieck trace. ?

GENERALIZED DELZANT DOMAINS
(the chambers)

$K \subset M$ codim. 0 submanifold with corners.

Near each codim 1 face f we have a smooth function $\mu_f : U_f \to \mathbb{R}$ which generates a free S^1-action with $f \subset \{\mu_f = 0\}$



$+$ grading data assumptions $+ \dots$

(free can be relaxed)

HOMOLOGICAL SECTION

$L \subset M$ Lagrangian is called a K-h.s. if:

it is the fixed point set of anti-symp. inv. $\varphi : M \to M$ with $\varphi(K) = K$ &

For every face $\bigcap_{i \in I} f_i \subset K$, (possibly codim 0)

$$\bigcap_{i \in I} U_{f_i} \longrightarrow \mathbb{R}^{|I|} \quad \text{(proper onto convex image)}$$

- L is transverse to each fibre and intersects each T^l orbit at most once
- Reduction Lagrangians are contractible. ...

LOCAL GENERATION

Def : $K \subset M$ compact, $L \subset M$ Lagrangian brane.

L satisfies generation criterion at K if

$$HH_*(CF^*_M(K;L;\Lambda)) \longrightarrow SH^{*+n}_M(K;\Lambda)$$

hits the unit.

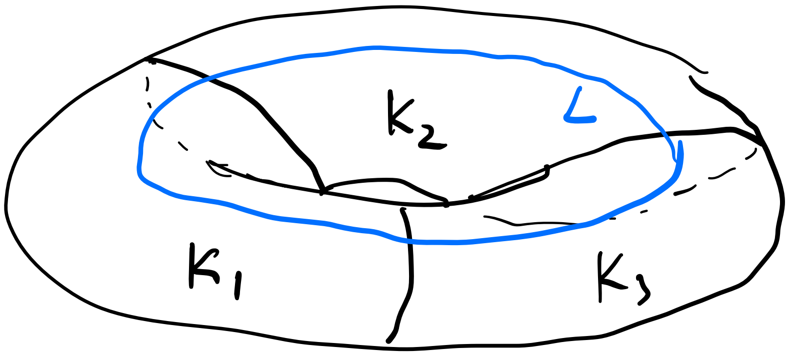
for K-homological section: homology concentrated in deg 0, affinoid algebra

Expectation : Ganatra's work for Liouville manifolds generalizes :

$$SC^*_M(K;\Lambda) \to CC^*(CF^*_M(K;L;\Lambda))$$

is a quasi-isomorphism of BV_∞-algebras.

BASIC EXAMPLE



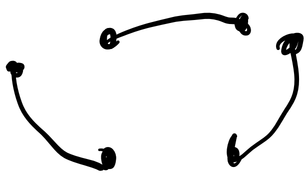
mirror

Glue non-archimedean annuli A_i of modulus $\text{Area}(K_i)$

Tate elliptic curve

$$\mathcal{Y} \cong \Lambda^* / q \sim T^{\text{area}(T^2)} \cdot q$$

$$\mathcal{Y} \xrightarrow{\text{trop}} \qquad \Lambda^*/q \xrightarrow{\text{val}} \mathbb{R}/\text{area}(T^2)\cdot\mathbb{Z}$$



HMS

$\text{Fuk}(M) \to \text{Coh}^{dg}(\mathcal{Y})$

(GAGA) $\cong \text{Coh}^{dg}(Y)$

Y is some algebraic elliptic curve $/\Lambda$