Stable Fukaya categories of Milnor fibers

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Abelian surface

- X: a principally polarized abelian surface
- $D \subset X$: the theta divisor (a curve of genus 2)
- $U := X \setminus D$: an affine manifold
- $\widetilde{\check{U}} \to \mathbb{P}^2$: the blow-up at three points in general position
- $\widetilde{\check{U}} \to \check{U}$: identification of the opposite sides of the toric momentum polytope of $\widetilde{\check{U}}$
- The toric boundary of $\widetilde{\check{U}}$ goes to the union of there rational curves whose intersection complex is a θ-shaped graph.
- $\operatorname{fuk} D \simeq \operatorname{scoh} \check{U} := \operatorname{coh} \check{U} / \operatorname{perf} \check{U}$ by Seidel.
- $\operatorname{wfuk} U \simeq \operatorname{coh} \check{U}$ and $\operatorname{fuk} U \simeq \operatorname{perf} \check{U}$, so that $\operatorname{wfuk} U / \operatorname{fuk} U \simeq \operatorname{fuk} D$?

Stable Fukaya category

- The *stable Fukaya category* is the quotient

$$\mathrm{sfuk}\, U := \mathrm{wfuk}\, U / \mathrm{fuk}\, U.$$

- According to Ganatra–Gao–Venkatesh, if there is a Koszul duality between wfuk U and fuk U, then the stable Fukaya category is equivalent to the *Rabinowitz Fukaya category*.

Balanced Fukaya category

- A symplectic manifold (M, ω) is *monotone* if $c_1(M) = k[\omega]$ for $k \neq 0$.
- Fix a 1-form θ on the principal S^1-bundle associated with the complex line bundle $\bigwedge^n(TM, J)$ such that $d\theta = \pi^*\omega$.
- A Lagrangian submanifold $L \subset M$ is *balanced* if $[i^*\theta] = 0 \in H^1(L; \mathbb{R})$.
- The balanced Fukaya category $\operatorname{fuk} M$ is decomposed into summands associated with eigenvalues of quantum multiplication by $c_1(M)$.
- The balanced Fukaya category is $\mathbb{Z}/2N\mathbb{Z}$-graded, where N is the minimal Maslov number.
- A $\mathbb{Z}/2N\mathbb{Z}$-graded category is the same as a $\mathbf{k}[u, u^{-1}]$-linear $\mathbb{Z}$-graded category for $\deg u = 2N$.

Complement of a smooth ample divisor

- X: a smooth projective n-fold
- $D \subset X$: a smooth ample divisor
- $U := X \setminus D$: an affine manifold
- Assume $rD \sim K_X$ for some $r \in \mathbb{Z}$, so that $c_1(U) = 0$.
- $\theta \in \mathrm{SH}^{2+2r}(U)$: the Borman–Sheridan–Varolgüneș class
- θ should be the first order deformation class of $\mathrm{fuk}(X, D)$.
- $\mathrm{fuk}\, U$ is proper over $\mathbf{k}[\theta]$ by Pomerleano.
- It is natural to expect $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{wfuk}\, U, \mathrm{perf}\, \mathbf{k}) \simeq \mathrm{fuk}\, U$, so that

$$\mathrm{sfuk}\, U \simeq \begin{cases} \mathrm{wfuk}\, U \otimes_{\mathbf{k}[\theta]} k(\theta) & \text{if } U \text{ is log Calabi–Yau,} \\ \mathrm{wfuk}\, U \otimes_{\mathbf{k}[\theta]} \mathbf{k}[\theta, \theta^{-1}] & \text{otherwise.} \end{cases}$$

Lagrangian correspondence

- $C \coloneqq \mathcal{SN}_{D/X}$: the sphere bundle of the normal bundle (the boundary of a tubular neighborhood)
- $C \to U \times D$: a Lagrangian correspondence
- C induces a functor $\Phi \colon \operatorname{wfuk} U \to \operatorname{fuk} D$.

Conjecture (Lekili–U).

If U is not log Calabi–Yau, then Φ descends to an equivalence

$$\Psi \colon \operatorname{sfuk} U \to \operatorname{fuk}(D; \lambda)$$

with the big summand of the balanced Fukaya category.

Cotangent bundle (with the canonical background)

- wfuk $T^*N \simeq \text{mod } C_{-\bullet}(\Omega N)$: the dg category of modules over chains on the based loop space equipped with the Pontryagin product (Abbondandolo–Schwarz, Fukaya–Seidel–Smith, Abouzaid).
- It is the global section of a locally constant cosheaf of dg categories (..., Kontsevich, ...).

Fukaya categories of T^*S^1

- $S^1 \simeq K(\mathbb{Z}, 1)$
- $\Omega S^1 \simeq \mathbb{Z}$
- $C_{-\bullet}(\Omega N) \simeq \mathbf{k}[\mathbb{Z}] \simeq \mathbf{k}\left[x, x^{-1}\right]$
- $\text{wfuk}\, T^*S^1 \simeq \text{mod}\, \mathbf{k}\left[x, x^{-1}\right] \simeq \text{coh}\, \mathbb{G}_{\mathrm{m}}$
- $\text{fuk}\, T^*S^1 \simeq \text{mod}_{\text{fin dim}}\, \mathbf{k}\left[x, x^{-1}\right] \simeq \text{coh}_{\text{cpt supp}}\, \mathbb{G}_{\mathrm{m}}$
- $\text{sfuk}\, T^*S^1 \simeq \text{mod}\, \mathbf{k}(x)$

Fukaya categories of T^*S^n for $n \geq 2$

- $C_{-\bullet}(\Omega S^n) \simeq \mathbf{k}[x]$
- $\deg x = 1 - n$
- wfuk $T^*S^n \simeq \operatorname{mod} \mathbf{k}[x]$
- fuk $T^*S^n \simeq \operatorname{mod}_{\text{fin dim}} \mathbf{k}[x]$
- $\mathbf{k}[x]$ is non-(super)commutative for even n.
- wfuk $T^*S^n \simeq \operatorname{coh} \mathbb{A}^1[n-1]$ and
  fuk $T^*S^n \simeq \operatorname{coh}_0 \mathbb{A}^1[n-1]$ for odd n.
- sfuk $T^*S^n \simeq \operatorname{mod} \mathbf{k}[x, x^{-1}]$

Brieskorn–Pham singularities

- A *Brieskorn–Pham polynomial* is a polynomial of the form $x_1^{a_1} + \cdots + x_n^{a_n}$ for $a_1, \ldots, a_n \in \mathbb{N}$.
- A *Brieskorn–Pham singularity* is an isolated hypersurface singularity at the origin of the affine complex hypersurface defined by a Brieskorn–Pham polynomial.
- The *Milnor fiber* is the Liouville domain obtained as the intersection of a regular level set with a ball.
- The contact boundary of the Milnor fiber of a Brieskorn–Pham singularity is the *Brieskorn manifold*.
- The Brieskorn 7-spheres with $(a_1, a_2, a_3, a_4, a_5) = (2, 2, 2, 3, 6k - 1)$ for $1 \leq k \leq 28$ give all 28 possible smooth structures on S^7.

Simple singularities

$$
\begin{aligned}
A_\ell &\colon x_1^{\ell+1} + x_2^2 + \cdots + x_{n+1}^2 = 0, \quad \ell = 1, 2, \ldots \\
D_\ell &\colon x_1^{\ell-1} + x_1 x_2^2 + x_3^2 + \cdots + x_{n+1}^2 = 0, \quad \ell = 4, 5, \ldots \\
E_6 &\colon x_1^4 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 = 0, \\
E_7 &\colon x_1^3 + x_1 x_2^3 + x_3^2 + \cdots + x_{n+1}^2 = 0, \\
E_8 &\colon x_1^5 + x_2^3 + x_3^2 + \cdots + x_{n+1}^2 = 0.
\end{aligned}
$$

- Q: a Dynkin quiver of the corresponding type
- $\mathscr{G}_Q$: the Ginzburg dg algebra of Q (without potential)
- $\mathscr{G}_Q$ is smooth and Koszul dual to the trivial extension algebra $B_Q := A_Q \oplus A_Q^\vee[-n]$ where A_Q is the path algebra of Q and $A_Q^\vee := \hom_{\mathbf{k}}(A_Q, \mathbf{k})$ (Keller).
- wfuk $U \simeq \operatorname{perf} \mathscr{G}_Q$ for $n \geq 2$ and char $\mathbf{k} = 0$ (Etgü–Lekili)

Cluster category

- $\mathscr{G}_Q$ is smooth but not proper.
- B_Q is proper but not smooth.
- $\mathscr{G}_Q$ and B_Q are Koszul dual.
- pseu B_Q: the category of pseudo-perfect B_Q-modules (dg modules over B_Q which are perfect as $\mathbf{k}$-modules)
- pseu $B_Q/$ perf $B_Q \simeq C_{n-1}(A_Q)$: the cluster category (the orbit category with respect to $\mathbb{S}[-n+1]$) (Keller)
- wfuk $U \simeq$ perf $\mathscr{G}_Q \simeq$ pseu B_Q
- fuk $U \simeq$ pseu $\mathscr{G}_Q \simeq$ perf B_Q
- sfuk $U \simeq C_{n-1}(A_Q)$

Trichotomy

- $U \coloneqq \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_1^{a_1} + \cdots + x_n^{a_n} = 1\}$ is
  - log Fano if $\frac{1}{a_1} + \cdots + \frac{1}{a_n} > 1$,
  - log Calabi–Yau if $\frac{1}{a_1} + \cdots + \frac{1}{a_n} = 1$, and
  - of log general type if $\frac{1}{a_1} + \cdots + \frac{1}{a_n} < 1$.
- Milnor fibers of simple singularities in dimension 2 are log Fano, although they admit complete Ricci-flat Kähler metric.
- Mirrors of log Calabi–Yau manifolds are log Calabi–Yau manifolds, and mirrors of others are Landau–Ginzburg models.

Koszul duality

- $U := \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_1^{a_1} + \cdots + x_n^{a_n} = 1\}$: the Milnor fiber of a Brieskorn–Pham singularity
- $(S_i)_{i=1}^{\mu}$: a distinguished basis of vanishing cycles in U
- $(L_i)_{i=1}^{\mu}$: non-compact Lagrangians such that $\dim \hom(L_i, S_j) = \delta_{ij}$
- $\mathscr{F} := \operatorname{end}\left(\bigoplus_{i=1}^{\mu} S_i\right), \ \mathscr{W} := \operatorname{end}\left(\bigoplus_{i=1}^{\mu} L_i\right)$
- If U is not log Calabi–Yau, then
  - $\mathscr{F} \simeq \hom_{\mathscr{W}}(\Bbbk, \Bbbk)$
  - $\mathscr{W}^{\mathrm{op}} \simeq \hom_{\mathscr{F}^{\mathrm{op}}}(\Bbbk, \Bbbk)$
  - $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{fuk}\, U, \mathrm{perf}\, \mathbf{k}) \simeq \mathrm{wfuk}\, U$
  - $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{wfuk}\, U, \mathrm{perf}\, \mathbf{k}) \simeq \mathrm{fuk}\, U$
  - $\mathrm{HH}^*(\mathrm{fuk}\, U) \simeq \mathrm{HH}^*(\mathrm{wfuk}\, U) \left(\overset{\text{Ganatra}}{\simeq} \mathrm{SH}^*(U) \right)$

Homological mirror symmetry

- $U \coloneqq \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_1^{a_1} + \cdots + x_n^{a_n} = 1\}$
- $S \coloneqq \mathbf{k}[x_0, \ldots, x_n]/(x_1^{a_1} + \cdots + x_n^{a_n} - x_0 \cdots x_n)$
- $K \coloneqq$ $\{(t_0, \ldots, t_n) \in (\mathbb{G}_{\mathrm{m}})^{n+1} \mid t_1^{a_1} = \cdots = t_n^{a_n} = t_0 \cdots t_n\}$
- $\check{U} \coloneqq [\operatorname{Spec} S/K]$: the quotient stack
- $\operatorname{scoh} \check{U} \coloneqq \operatorname{coh} \check{U}/\operatorname{perf} \check{U}$: the stable derived category (a.k.a. the singularity category)
- If U is not log Calabi–Yau, then wfuk $U \simeq$ scoh $\check{U}$.

CY/LG correspondence in the log Calabi–Yau case

- $K := \{(t_1, \dots, t_n) \in (\mathbb{G}_\mathrm{m})^n \mid t_1^{a_1} = \cdots = t_n^{a_n}\}$
- $\mathbb{X} := \left[(\mathbb{A}^n \setminus \mathbf{0}) \,/\, K \right]$
- $\mathbb{D}_0 := V(x_1^{a_1} + \cdots + x_n^{a_n})$, $\mathbb{D}_\infty := V(x_1 \cdots x_n)$.
- $\widetilde{\mathbb{X}} \to \mathbb{X}$: the blow-up along $\mathbb{D}_0 \cap \mathbb{D}_\infty$
- $\check{\mathbb{U}} := \widetilde{\mathbb{X}} \setminus$ (the strict transform of $\mathbb{D}_\infty$) : smooth
- scoh $\check{U} \simeq \operatorname{coh} \check{\mathbb{U}}$ by a variation of Orlov's theorem

Stable derived categories

- $S := \mathbf{k}[x_0, \ldots, x_n]/(x_1^{a_1} + \cdots + x_n^{a_n} - x_0 \cdots x_n)$
- $K :=$
  $\{(t_0, \ldots, t_n) \in (\mathbb{G}_{\mathrm{m}})^{n+1} \mid t_1^{a_1} = \cdots = t_n^{a_n} = t_0 \cdots t_n\}$
- $\check{U} := [\operatorname{Spec} S/K]$
- $\check{E} := \{x_0 = 0\} \subset \check{U}$
- $\check{D} := \check{U} \setminus \check{E}$
- If U is not log CY, then $\check{D} \simeq [\operatorname{Spec} T/H]$ where
  $T := \mathbf{k}[x_1, \ldots, x_n]/(x_1^{a_1} + \cdots + x_n^{a_n} - x_1 \cdots x_n)$ and
  $H := K \cap \{t_0 = 1\}$.
- $\operatorname{coh} \check{D} \simeq \operatorname{coh} \check{U} / \operatorname{coh}_{\check{E}} \check{U}$
- $\operatorname{scoh} \check{D} \simeq \operatorname{scoh} \check{U} / \operatorname{scoh}_{\check{E}} \check{U}$
- If U is not log CY, then $\operatorname{scoh}_{\check{E}} \check{U}$ is mirror to $\operatorname{fuk} U$ inside $\operatorname{wfuk} U \simeq \operatorname{scoh} \check{U}$, so that $\operatorname{sfuk} U \simeq \operatorname{scoh} \check{D}$.

$\mathbb{P}^n \setminus$ (a smooth anticanonical divisor)

- $D := \left\{ [x_0 : \cdots : x_n] \in \mathbb{P}^n \,\middle|\, x_0^{n+1} + \cdots + x_{n+1}^{n+1} = 0 \right\}$
- $\widetilde{U} := \left\{ (x_0, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \,\middle|\, x_0^{n+1} + \cdots + x_{n+1}^{n+1} = 1 \right\}$
- $\boldsymbol{\mu}_{n+1}$ acts diagonally on $\widetilde{U}$.
- $\widetilde{U}/\boldsymbol{\mu}_{n+1} \cong U := \mathbb{P}^n \setminus D$
- $G :=$ $\left\{ [\operatorname{diag}(t_0, \ldots, t_n)] \in \mathrm{PSL}_{n+1} \,\middle|\, t_0^{n+1} = \cdots = t_n^{n+1} = 1 \right\}$
- $\mathbb{D}_0 := \left\{ x_0^{n+1} + \cdots + x_n^{n+1} = 0 \right\} \subset \mathbb{P} := [\mathbb{P}^n / G]$
- $\mathbb{D}_\infty := \{ x_0 \cdots x_n = 0 \} \subset \mathbb{P}$
- $\widetilde{\mathbb{P}} \to \mathbb{P}$: the blow-up along $\mathbb{D}_0 \cap \mathbb{D}_\infty$
- $\widetilde{\mathbb{D}}_\infty \subset \widetilde{\mathbb{P}}$: the strict transform of $\mathbb{D}_\infty$
- $\check{\mathbb{U}} := \widetilde{\mathbb{P}} \setminus \widetilde{\mathbb{D}}_\infty$
- wfuk $U \simeq$ coh $\check{\mathbb{U}}$?

$\mathbb{P}^2 \setminus$ (an anticanonical divisor)

- If $n = 2$, then $G \cong \boldsymbol{\mu}_3$ and the minimal resolution $X \to \mathbb{P}^2/\boldsymbol{\mu}_3$ is a toric weak del Pezzo surface of degree 3.
- The strict transform D_∞ of $\mathbb{D}_\infty$ is the toric boundary of X consisting of three (-1)-curves and six (-2)-curves.
- The blow-up $\widetilde{X} \to X$ is a rational elliptic surface.
- The strict transform $\widetilde{D}_\infty$ of D_∞ is an anticanonical cycle consisting of nine (-2)-curves, which is a singular fiber of an elliptic fibration $\widetilde{X} \to \mathbb{P}^1$.
- $\check{U} := \widetilde{X} \setminus \widetilde{D}_\infty$ is mirror to the complement of a smooth anticanonical divisor in $\mathbb{P}^2$.
- $\check{U}$ is projective over $\mathbb{A}^1$, whose coordinate should be mirror to the BSV class.
- In contrast, $\mathbb{P}^2$ minus a conic and a line in general position is self-mirror, and hence affine in particular.
- $\mathbb{P}^2$ minus three lines in general position is also self-mirror.