Monodromy of GKZ Hypergeometric functions and homological mirror symmetry.

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Homological mirror symmetry

Conjecture: $\forall$ Calabi-Yau variety Y, $\exists$ Calabi-Yau var. X s.t.

$$D^b\mathfrak{F}uk(Y) \cong D^b coh\, X,$$

i.e. the equivalence between the derived category of Fukaya category of Y ($\Rightarrow$ module of vanishing cycles of Y) and the derived category of coherent sheaves of X as enhanced triangulated categories.
Consequence:

$$Auteq(D^b\mathfrak{F}uk(Y)) \cong Auteq(D^b coh\, X).$$

An isomorphism between two self-equivalence groups holds.

Our aim

To realize the essential part (Grothendieck group level) of $Auteq(D^b\mathfrak{F}uk(Y))$, i.e. $Aut(H_*(Y), \mathbf{C})$ as the global monodromy of $Y_s, s \in \mathbf{C}^N \setminus Discriminant$ with the aid of the Todd class $Todd_X$ of the tangent bundle TX,

$$Mon : \pi_1(\mathbf{C}^N \setminus Discriminant) \longrightarrow GL(H_*(Y), \mathbf{C}).$$

Monodromy of the period integrals for $Y_{\mathbf{s}}$:

$$\Psi(\mathbf{s}, \boldsymbol{\lambda}) \to \Psi(\mathbf{s}, \boldsymbol{\lambda}) - \int_X Todd_X([\mathbf{D}])\Psi(s, [\mathbf{D}]/2\pi i),$$

with $[\mathbf{D}] \in H^2_{toric}(X)$. Recall Riemann-Roch-Hirzebruch Theorem. (Kontsevich proposal 1998)

Global monodromy of Gauss HGF

$F(\alpha,\beta,\gamma|s)$ determined by analytic continuation of the series

$$F(\alpha,\beta,\gamma|s)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{m\geq 0}\frac{\Gamma(\alpha+m)\Gamma(\beta+m)}{\Gamma(\gamma+m)\Gamma(1+m)}s^m$$

for $\alpha,\beta,\gamma\notin\mathbb{Z}_{\leq 0}$. Convergent for $|s|<1$
For $|s|<1$, $\alpha,\beta,\gamma\in\mathbb{Q}$, $\Re\gamma>\Re\beta>0$,

$$F(\alpha,\beta,\gamma|s)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xs)^{-\alpha}dx$$

Period of a curve

$$y^N=x^{m_1}(1-x)^{m_2}(1-xs)^{m_3}.$$

Mellin-Barnes integral 1

Mellin-Barnes integral for $F(\alpha, \beta, \gamma|s)$

$$\frac{1}{2\pi i} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\alpha+z)\Gamma(\beta+z)\Gamma(-z)}{\Gamma(\gamma+z)} exp(\pi iz)\phi(z)s^z dz \tag{3.1}$$

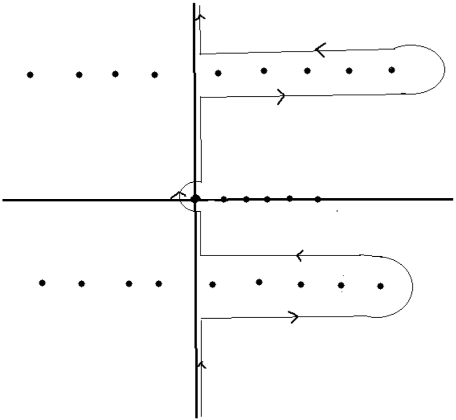
where $\int_{c-i\infty}^{c+i\infty}$: contour located left to all non-negative integers and right to other poles of the integrand. ϕ : meromorphic function $\phi(z+1) = \phi(z)$.

$$Res_{z=m}\Gamma(-z) = \frac{(-1)^{m+1}}{m!}.$$

(3.1) solutions to

$$[\theta_s(\theta_s+\gamma-1) - s(\theta_s+\alpha)(\theta_s+\beta)]u(s) = 0, \quad \theta_s = s\frac{\partial}{\partial s}. \tag{3.2}$$

Mellin-Barnes integration contour



Mellin-Barnes integral representations for HGF

$$\varphi_1(z) = \frac{\Gamma(z+\alpha)\Gamma(z+\beta)}{\Gamma(z+\gamma)\Gamma(z+1)} \frac{e^{-\pi i z} s^z}{\sin \pi z}$$

$$\varphi_2(z) = \frac{\Gamma(z+\alpha)\Gamma(z+\beta)}{\Gamma(z+\gamma)\Gamma(z+1)} \frac{e^{-\pi i (z+\gamma-1)} s^z}{\sin \pi (z+\gamma-1)}$$

$$y_1^*(s) = Res_{z \in \mathbb{Z}_{\geq 0}} \varphi_1(z) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\pi \Gamma(\gamma)} F(\alpha, \beta, \gamma | s)$$

$$y_2^*(s) = Res_{z \in -\gamma+1+\mathbb{Z}_{\geq 0}} \varphi_2(z)$$
$$= e^{-\pi i (\gamma-1)} \frac{\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)}{\pi \Gamma(2-\gamma)}$$
$$s^{-\gamma+1} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma | s)$$

Mellin-Barnes integral: contour throw

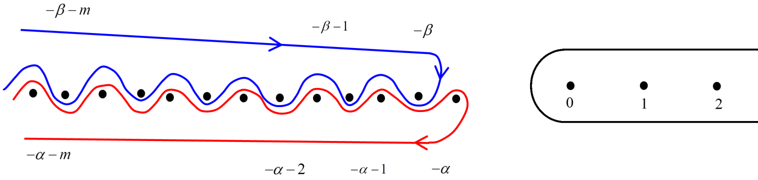


Figure: Mellin-Barnes contour throw for Gauss HGF.

Connection formula between local solutions to Gauss HG Eqn.

Notation:$e(\lambda) = e^{2\pi i \lambda}$

$$(s \sim 0) \quad y_1^*(s) =$$

$$e^{-\pi i(1-\gamma+\alpha-\beta)}\left(\frac{e(\alpha+1-\gamma)}{e(\beta-\alpha)-1}\right)\bar{y}_1^*(s)$$

$$+e^{-\pi i(1-\gamma-\alpha+\beta)}\left(\frac{e(\beta+1-\gamma)}{e(\alpha-\beta)-1}\right)\bar{y}_2^*(s) \quad (s \sim \infty)$$

$$\bar{y}_1^*(s) = Res_{z \in -\alpha+\mathbb{Z}_{\leq 0}}\varphi_1(z)$$

$$= \sum_{m \geq 0} \frac{\Gamma(\alpha+m)\Gamma(\alpha+1-\gamma+m)}{\Gamma(1-\beta+\alpha+m)\Gamma(1+m)} s^{-\alpha-m} \quad (s \sim \infty)$$

$$\bar{y}_2^*(s) = Res_{z \in -\beta+\mathbb{Z}_{\leq 0}}\varphi_1(z) \quad (s \sim \infty)$$

Connection formula between local solutions to Gauss HG Eqn.,

$$(s \sim 0) \quad \begin{pmatrix} y_1^*(s) \\ y_2^*(s) \end{pmatrix} = e^{\pi i(1-\gamma-\alpha-\beta)} PQ \begin{pmatrix} \bar{y}_1^*(s) \\ \bar{y}_2^*(s) \end{pmatrix} \quad (s \sim \infty)$$

where the matrices P and Q are given by

$$P = \begin{pmatrix} \frac{1-e(-\alpha-1+\gamma)}{1-e(-\alpha)} & \frac{1-e(-\beta-1+\gamma)}{1-e(-\beta)} \end{pmatrix}.$$

$$Q = diag\left(\frac{e(2\alpha)}{1-e(\alpha-\beta)}, \frac{e(2\beta)}{1-e(\beta-\alpha)}\right).$$

$C = e^{\pi i(1-\gamma-\alpha-\beta)} PQ$: connection matrix.

Monodromy via Mellin-Barnes integral

Monodromy of the solution basis $(y_1^*(s), y_2^*(s))$ to Gauss HG eqn.

$$h_0 = \rho(\gamma_0) = \begin{pmatrix} 1 & 0 \\ 0 & e(-\gamma) \end{pmatrix},$$

$$h_\infty = \rho(\gamma_\infty) = C \begin{pmatrix} e(-\alpha) & 0 \\ 0 & e(-\beta) \end{pmatrix} C^{-1},$$

$C = e^{\pi i(1-\gamma-\alpha-\beta)} PQ$: connection matrix.
Monodromy around $s = 1$, $h_1 = (h_\infty h_0)^{-1}$.

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to GL(2, \mathbb{C})$$

Global monodromy group:

$$< h_0, h_1, h_\infty >=< h_0, h_\infty > .$$

Newton polyhedron of an affine C.Y. hypersurface 1

Laurent polynomial with deformation parameter coefficients $\mathbf{a} := (a_1, \cdots, a_N) \in \mathbf{T}^N = (\mathbb{C}^*)^N$,

$$F(x, x_n, \mathbf{a}) = x_n(a_1 + a_2 x^{\alpha_2} + \cdots + a_N x^{\alpha_N}). \qquad (4.1)$$

s.t. $F(x, x_n, \mathbf{a}) \in \mathbb{C}[x^{\pm}][x_n, \mathbf{a}]$ where $x^{\pm} = (x_1^{\pm}, \cdots, x_{n-1}^{\pm})$, $\{\alpha_j\}_{j=1}^{N} \subset \mathbb{Z}^{n-1}$. Here $\alpha_1 = 0 \in \mathbb{Z}^{n-1}$, $\alpha_1 \in \Delta(F)^{int}$ $F(x, 1, \mathbf{1})$,

$$\{0\} \in \Delta(F) = \text{convex hull of } \{\alpha_j\}_{j=1}^{N} \subset \mathbb{R}^{n-1}. \qquad (4.2)$$

$$\bar{\Delta}(F) = \text{convex hull of } \{\bar{\alpha}_p\}_{p=1}^{N} \cup \{0\} \subset \mathbb{R}^n. \quad (4.3)$$

for

$$\bar{\alpha}_p = \begin{pmatrix} \alpha_p \\ 1 \end{pmatrix}.$$

This $n-$dimensional polyhedron is the Newton polyhedron of

$$F(x, x_n, \mathbf{1}) + 1 = x_n f(x) + 1 \quad (4.4)$$

Associate to (4.1) a $n \times N$ matrix A,

$$\mathrm{A} = \left(\bar{\alpha}_1, \quad \cdots, \quad \bar{\alpha}_N \right). \quad (4.5)$$

$n \times N$ matrix A,

$$A = \left(\ \bar{\alpha}_1, \quad \cdots, \quad \bar{\alpha}_N \ \right). \qquad (4.6)$$

Condition on A ($\Rightarrow$ Gorenstein cone):

$$\mathbb{Z}^n = \sum_{p=1}^{N} \mathbb{Z}\bar{\alpha}_p.$$

For $\bar{\Delta} = \bar{\Delta}(F)$

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbb{Z}^n & \rightarrow & \mathbb{Z}^{\Sigma(\bar{\Delta})(1)} & \rightarrow & A_{n-1}(X_{\Sigma(\bar{\Delta})}) & \rightarrow & 0 \\ & & \mathbf{m} & \mapsto & (< \mathbf{m}, \bar{\alpha}_p >)_{p=1}^N & & & & \\ & & & & (n_p)_{p=1}^N & \mapsto & \sum_{p=1}^N n_p D_p & & \end{array}$$

with $A_{n-1}(X_{\Sigma(\bar{\Delta})})$ the Chow group of rank $d := N - n$ of the toric variety $X_{\Sigma(\bar{\Delta})}$.

Gale transform of A (1)

Lattice $\mathbb{L} \subset \mathbb{Z}^N$ generated by $d = N - n$ integer vectors,

$$\ell_1^{(j)} \bar{\alpha}_1 + \cdots + \ell_N^{(j)} \bar{\alpha}_N = 0, \quad j \in [1; d].$$

$$\mathbb{L} = \oplus_{j=1}^{d} \mathbb{Z} \vec{\ell}^{(j)} \subset \mathbb{Z}^N, \tag{4.7}$$

where

$$B = \begin{pmatrix} \vec{\ell}^{(1)} \\ \vdots \\ \vec{\ell}^{(d)} \end{pmatrix} = (\mathfrak{b}_1, \cdots, \mathfrak{b}_N) \tag{4.8}$$

B: a (specially chosen) Gale transform of the $N \times n$ matrix A i.e. $\vec{\ell}^{(j)}$, $j \in [1; d]$ are orthogonal to the rows of A.

$$\vec{\ell}^{(j)} := (\ell_1^{(j)}, \cdots, \ell_N^{(j)}), \ j \in [1; d],$$

$$\mathfrak{b}_p := (\ell_p^{(1)}, \cdots, \ell_p^{(d)})^t, \ p \in [1; N].$$

Gale transform of A (2)

$$B = \begin{pmatrix} \vec{\ell}^{(1)} \\ \vdots \\ \vec{\ell}^{(d)} \end{pmatrix} = (\mathfrak{b}_1, \cdots, \mathfrak{b}_N) \qquad (4.9)$$

B: a (specially chosen) Gale transform of the $N \times n$ matrix A i.e. $\vec{\ell}^{(j)}$, $j \in [1; d]$ are orthogonal to the rows of A.

$$\vec{\ell}^{(j)} := (\ell_1^{(j)}, \cdots, \ell_N^{(j)}),\ j \in [1; d],$$

For every $j \in [1; d]$ define

$$\begin{aligned} I_-^{(j)} &= \{p \in [1; N]; \ell_p^{(j)} < 0\} \\ I_+^{(j)} &= \{p \in [1; N]; \ell_p^{(j)} > 0\} \qquad (4.10) \\ I_0^{(j)} &= \{p \in [1; N]; \ell_p^{(j)} = 0\}. \end{aligned}$$

$$1 \to \mathbf{T}^n \to \mathbf{T}^N \xrightarrow{exp^{\mathrm{B}}} \mathbf{T}^d \to 1. \qquad (4.11)$$

where

$$\mathrm{B} log\ \mathbf{a} = log\ \mathbf{s}$$

for $\mathbf{s} = exp^{\mathrm{B}}(\mathbf{a}) \in \mathbf{T}^d$ and $\mathbf{a} \in \mathbf{T}^N$, $N = |\Sigma(\bar{\Delta})(1)|$.
Iintroduce a deformation

$$f(x, x_n, \mathbf{s}) = x_n\Big(\sum_{j \in \mathcal{J}} s_j x^{\alpha_j} + \sum_{\bar{j} \notin \mathcal{J}} x^{\alpha_{\bar{j}}}\Big). \qquad (4.12)$$

with $|\mathcal{J}| = d$.

Example, $n = 4$.

$$f(x, x_4, \mathbf{s}) = x_4(1 + x_1 + x_2 + \frac{s_1}{x_1 x_2} + x_3 + \frac{s_2}{x_3})$$

or

$$F(x, x_4, \mathbf{a}) = x_4(a_1 + a_2 x_1 + a_3 x_2 + \frac{a_4}{x_1 x_2} + a_5 x_3 + \frac{a_6}{x_3})$$

defining the affine part of a bi-degree $(3, 2)$ K3 surface in $\mathbb{P}^2 \times \mathbb{P}^1$.

$$N = 6, d = N - n = 2.$$

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}.$$

$$B = \begin{pmatrix} -3 & 1 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} s_1 \\ s_2 \end{matrix}$$

$$= \begin{pmatrix} \mathfrak{b}_1 & \mathfrak{b}_2 & \mathfrak{b}_3 & \mathfrak{b}_4 & \mathfrak{b}_5 & \mathfrak{b}_6 \end{pmatrix} = \begin{pmatrix} \vec{\ell}^{(1)} \\ \vec{\ell}^{(2)} \end{pmatrix}$$

$$I_+^{(1)} = \{2,3,4\}, I_-^{(1)} = \{1\}, I_0^{(1)} = \{5,6\}.$$

$$I_+^{(2)} = \{5,6\}, I_-^{(1)} = \{1\}, I_0^{(1)} = \{2,3,4\}.$$

The parameter transition:

$$s_1 = \frac{a_2 a_3 a_4}{a_1^3}, \quad s_2 = \frac{a_5 a_6}{a_1^2}.$$

GKZ $A-$ hypergeometric function

Residue along

$$Y_{\mathbf{a}} = \{x \in \mathbf{T}^{n-1}; F(x,1,\mathbf{a}) = a_1 x^{\alpha_1} + \cdots + a_{N-1} x^{\alpha_{N-1}} + a_N = 0\},$$

$$\Phi_{\gamma_{\mathbf{a}}}(\mathbf{a}) := \int_{\mathfrak{t}(\gamma_{\mathbf{a}})} F(x,1,\mathbf{a})^{-1} \frac{dx}{x^{\mathbf{1}}}, \tag{4.13}$$

$\mathfrak{t}(\gamma_{\mathbf{a}}) \in H_{n-1}(\mathbf{T}^{n-1} \setminus Y_{\mathbf{a}})$: Leray's coboundary.
Notations
$\mathbf{z} = (z_1, \cdots, z_d)$,
$\mathbf{s} = (s_1, \cdots, s_d)$,

Proposition

1) The GKZ A−HGF $\Phi_{\gamma_{\mathbf{a}}}(\mathbf{a}) \in$ *ker (A-GKZ HGS)*

$$(\prod_{p\in I_+^{(j)}}(\frac{\partial}{\partial a_p})^{\ell_p^{(j)}} - \prod_{p\in I_-^{(j)}}(\frac{\partial}{\partial a_i})^{-\ell_p^{(j)}})\Phi(\mathbf{a}) = 0, \quad j \in [1;d],$$

where $\mathbb{L} = \oplus_{j=1}^{d}\mathbb{Z}\vec{\ell}^{(j)}$ *(4.7).*

$$\sum_{p=1}^{N}\alpha_p a_p \frac{\partial}{\partial a_p}\Phi(\mathbf{a}) = 0 \ \ (weighted\ homogeneous\ of\ degree = 0).$$

$$\sum_{p=1}^{N} a_p \frac{\partial}{\partial a_p}\Phi(\mathbf{a}) = -\Phi(\mathbf{a}) \ \ (w.\ homog.\ of\ degree = -1).$$

2) $dim.\ ker\ (A - GKZ\ HGS) = (n-1)!vol_{n-1}\Delta(F).$

Mellin-Barnes integral representation

Proposition

For $Y_{\mathbf{s}} := \{x \in \mathbf{T}^{n-1}; f(x,1,\mathbf{s}) = \sum_{j\in\mathcal{J}} s_j x^{\alpha_j} + \sum_{\bar{j}\notin\mathcal{J}} x^{\alpha_{\bar{j}}} = 0\}$, $|\mathcal{J}| = d = N - n$
period integral

$$\tilde{\Phi}_\gamma(\mathbf{s}) := \int_{\mathfrak{t}(\gamma)} f(x,1,\mathbf{s})^{-1} \frac{dx}{x^{\mathbf{1}}}, \tag{4.14}$$

for $\mathfrak{t}(\gamma) \in H_{n-1}(\mathbf{T}^n \setminus Y_{\mathbf{s}})$.

Mellin-Barnes integral : multiple power series convergent in an open $\mathcal{V}_\rho \subset \mathbb{C}^d$:$\tilde{\Phi}^{(\rho)}_\gamma(\mathbf{s}) =$

$$\sum_{\tilde{\mathbf{z}}\in P_\rho} Res_{\mathbf{z}=\tilde{\mathbf{z}}}\Gamma(1- < \mathfrak{b}_1, \mathbf{z} >) \prod_{2\leq p\leq N} \Gamma(- < \mathfrak{b}_p, \mathbf{z} >)\varphi_\gamma(\mathbf{z})\mathbf{s}^{\mathbf{z}},$$

where $\varphi_\gamma(\mathbf{z})$: periodic $\varphi_\gamma(\mathbf{z}+\mathbf{z}_0) = \varphi_\gamma(\mathbf{z})$ $\forall \mathbf{z}_0 \in \mathbb{Z}^d$

Define

$$\Phi_{\gamma}^{(\rho)}(\mathbf{s}) = \sum_{\tilde{\mathbf{z}} \in P_\rho} Res_{\mathbf{z}=\tilde{\mathbf{z}}} \prod_{1 \leq p \leq N} \Gamma(- < \mathfrak{b}_p, \mathbf{z} >) \varphi_\gamma(\mathbf{z}) \mathbf{s}^{\mathbf{z}}, \tag{4.15}$$

Suffix $\rho \in [1; Q]$

$\leftrightarrow \mathcal{T}_\rho$ regular triangulation.

$\leftrightarrow$ vertex of the secondary polytope of A

$\leftrightarrow P_\rho$: support of the power series $\leftrightarrow$ *cone* $-C_\rho^\vee$

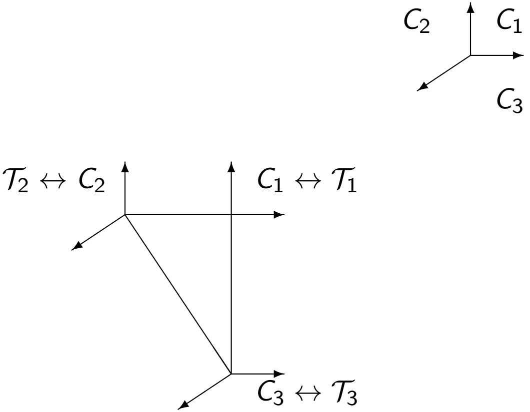
$\leftrightarrow \mathcal{V}_\rho$ domain of convergence $\leftrightarrow$ *cone* C_ρ .

Here P_ρ :

$$- < \mathfrak{b}_p, \mathbf{z} > \in \mathbb{Z}_{\leq 0} \text{ for } p \in \mathcal{J}_\rho \subset [1; N] \tag{4.16}$$

where $\mathcal{J}_\rho \subset [1; N]$, $|\mathcal{J}_\rho| = rank(\mathfrak{b}_p)_{p \in \mathcal{J}_\rho} = d$.

Secondary polytope, secondary fan



$\Phi_{\gamma}^{(\rho)}(\mathbf{s})$ satisfies HG system of Horn type,
$\left(\prod_{p\in I_{+}^{(j)}}(-<\mathfrak{b}_p,\boldsymbol{\theta}_s>)_{\ell_p^{(j)}} - s_j \prod_{p\in I_{-}^{(j)}}(-<\mathfrak{b}_p,\boldsymbol{\theta}_s>)_{-\ell_p^{(j)}}\right)\Phi_{\gamma}^{(\rho)}$
$= 0, \quad \forall j \in [1;d]$, where

$$(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1),$$

the Pochhammer symbol.

$$\boldsymbol{\theta}_s = (s_1\frac{\partial}{\partial s_1},\cdots,s_d\frac{\partial}{\partial s_d}).$$

Example, n=4. Continuation

$$\Phi_\gamma(\mathbf{s}) = \sum_{\tilde{\mathbf{z}} \in P_1} Res_{\mathbf{z}=\tilde{\mathbf{z}}} \Gamma(3z_1+2z_2)\Gamma(-z_1)^3\Gamma(-z_2)^2\varphi_\gamma(\mathbf{z})\mathbf{s}^{\mathbf{z}} d\mathbf{z}.$$

e.g.

$$\varphi_{\gamma_0}(z) = \left(\frac{1-e(z_1)}{2\pi i}\right)^2 \left(\frac{1-e(z_2)}{2\pi i}\right), \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\ \pi z}.$$

s.t.

$\Phi_{\gamma_0}(\mathbf{s}) =$

$\sum_{\tilde{\mathbf{z}} \in P_1} Res_{\mathbf{z}=\tilde{\mathbf{z}}} \frac{\Gamma(3z_1+2z_2)\Gamma(-z_1)\Gamma(-z_2)}{\Gamma(z_1+1)^2\Gamma(z_2+1)} (e^{2\pi i}s_1)^{z_1}(e^{\pi i}s_2)^{z_2} dz_1 dz_2$

holomorphic near $(s_1, s_2) = (0, 0)$ for $P_1 = (\mathbb{Z}_{\geq 0})^2$.

Dimension ker (GKZ A-HGS) $= 6 = 3! vol(\Delta(F))$.

Definition

(Stanley-Reisner ring) Convex polyhedron $\bar{\Delta} \subset \mathbb{R}^n$ *:convex hull of*

$$A = \begin{pmatrix} \bar{\alpha}_1 & \cdots & \bar{\alpha}_N \end{pmatrix}$$

triangulation $\mathcal{T}$ *of* $\bar{\Delta}$ *define the Stanley-Reisner ring for* $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_N)$,

$$\mathcal{R}_{A,\mathcal{T}} := \mathbb{Z}[\boldsymbol{\mu}]/(\mathcal{I}_{lin} + \mathcal{I}_{mon}), \tag{5.1}$$

- $\mathcal{I}_{lin} = \left\langle \sum_{i=1}^N < u^\vee, \bar{\alpha}_i > \mu_i \right\rangle, \ \forall u^\vee \in (\mathbb{Z}^n)^\vee.$
- $\mathcal{I}_{mon} = \langle \mu_{i_1} \cdot \mu_{i_2} \cdots \mu_{i_s} \rangle$ *for convex hull* $\{\bar{\alpha}_{i_1}, \cdots, \bar{\alpha}_{i_s}\}$ *not a simplex in* $\mathcal{T}$.

$\mathbb{Q}[\boldsymbol{\mu}]/\mathcal{I}_{lin} \cong \mathbb{Q}[\boldsymbol{\lambda}]$ with $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_d)$ in such a way that

$$\mathcal{R}_{A,\mathcal{T}} \otimes \mathbb{Q} \cong \mathbb{Q}[\boldsymbol{\lambda}]/\tilde{\mathcal{I}}_{mon} \tag{5.2}$$

The ideal $\tilde{\mathcal{I}}_{mon}$ in (5.2) can be written

$$\tilde{\mathcal{I}}_{mon} = \left\langle \prod_{p \in I_+^{(1)}} < \mathfrak{b}_p, \boldsymbol{\lambda} >, \cdots, \prod_{p \in I_+^{(d)}} < \mathfrak{b}_p, \boldsymbol{\lambda} > \right\rangle . \tag{5.3}$$

for $I_+^{(j)} = \{p \in [1; N]; \ell_p^{(j)} > 0\}$.

Definition

The ideal $\mathcal{I}_{core}$ of $\mathbb{Z}[\boldsymbol{\mu}]$ is defined as a principal ideal generated by a monomial

$$\mu_{core} := \prod_{p \in \cap_j I_-^{(j)}} \mu_p .$$

We define

$$\bar{\mathcal{R}}_{\mathsf{A}} := \mathcal{R}_{\mathsf{A}} / Ann(\mathcal{I}_{core}).$$

In view of (5.2)

$$\bar{\mathcal{R}}_{\mathsf{A}} \otimes \mathbb{Q} \cong \tilde{\lambda}_{core} \cdot \mathbb{Q}[\boldsymbol{\lambda}] / \tilde{\mathcal{I}}_{mon}$$

for

$$\tilde{\lambda}_{core} = \prod_{p \in \cap_j I^{(j)}} < \mathfrak{b}_p, \boldsymbol{\lambda} > .$$

Definition

The cone

$$\Lambda = \sum_{p=1}^{N} \mathbb{R}_{\geq 0}\bar{\alpha}_p$$

is called **Gorenstein** *if*

(1) $\sum_{p=1}^{N} \mathbb{Z}\bar{\alpha}_p = \mathbb{Z}^n$.

(2) $\exists \alpha_0^\vee \in (\mathbb{Z}^n)^\vee$ *s.t.* $< \alpha_0^\vee, \bar{\alpha}_p >= 1, \forall p \in [1, N]$.

A Gorenstein cone is called **reflexive** *if its dual cone is also Gorenstein*

$$\Lambda^\vee = \{\beta \in (\mathbb{R}^n)^\vee; < \alpha, \beta > \geq 0, \forall \alpha \in \Lambda\}.$$

Basis of GKZ $A-$ HG solutions

Theorem

(J.Stienstra)
(1) The cone Λ defined by the matrix A be Gorenstein.

$\exists iso : Hom(\mathcal{R}_{\mathrm{A}}, \mathbb{C}) \cong$ *sol (GKZ A− HGS) with dimension*$= (n-1)! vol_{n-1}\Delta(F)$.

$$\exists inj : Hom(\bar{\mathcal{R}}_{\mathrm{A}}, \mathbb{C}) \hookrightarrow \bar{\mathcal{R}}_{\mathrm{A}}.$$

$$\bar{\mathcal{R}}_{\mathrm{A}} := \mathcal{R}_{\mathrm{A}} / Ann(\prod_{p \in \cap_j I_{-}^{(j)}} < \mathfrak{b}_p, \boldsymbol{\lambda} >).$$

(2) If the the cone Λ is **reflexive Gorenstein**($+$ natural conditions on $\tilde{\lambda}_{core}$), we have

$$\mathcal{R}_{\mathrm{A}} \otimes \mathbb{C} \cong H^*(X_{\Sigma(\Delta)}, \mathbb{C}),$$

with $X_{\Sigma(\Delta)}$: smooth projective toric variety.

$$\bar{\mathcal{R}}_{\mathrm{A}} = \mathcal{R}_{\mathrm{A}}/Ann(\mathcal{I}_{core}) \cong H^*_{toric}(W, \mathbb{Z})$$
$$:= image\ (H^*(X_{\Sigma(\Delta)}, \mathbb{Z}) \to H^*(W, \mathbb{Z})),$$

where W : a Calabi-Yau hypersurface in $X_{\Sigma(\Delta)}$ constructed by the polar polyhedron

$$\Delta(F)^* := \{\beta \in (\mathbb{R}^{n-1})^\vee; < \beta, \alpha > \geq -1,\ \forall \alpha \in \Delta(F)\}.$$

$Hom(\bar{\mathcal{R}}_{\mathrm{A}}, \mathbb{C})$: period integrals of $\bar{Y}_s \Longrightarrow$ Picard-Fuchs equation subtracted from GKZ HG system.

Example $n = 4$ continuation

$$\mathcal{R}_A = \mathbb{Z}[\mu]/(\mathcal{I}_{lin} + \mathcal{I}_{mon}) \cong \mathbb{Z}[\mu_4, \mu_6]/\left\langle \mu_4^3, \mu_6^2 \right\rangle$$

$\cong \sum_{(j,k)\in[0,2]\times[0,1]} \mathbb{Z}{\lambda_1}^j{\lambda_2}^k \cong H^*(\mathbb{P}^2 \times \mathbb{P}^1),$
$rank = 6 = 3! vol(\Delta(F)).$

$$\mathcal{I}_{lin} = \left\langle \sum_{p=1}^{6} \mu_p, \mu_2 - \mu_4, \mu_3 - \mu_4, \mu_5 - \mu_6 \right\rangle \text{ see A}$$

$$\mathcal{I}_{mon} = \langle \mu_2\mu_3\mu_4, \mu_5\mu_6 \rangle \text{ see B}$$

$\bar{\mathcal{R}}_A \cong \sum_{(j,k)\in[0,2]\times[0,1]} \mathbb{Z}{\lambda_1}^j{\lambda_2}^k / Ann(-3\lambda_1 - 2\lambda_2)$
$\cong \mathbb{Z} \oplus \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \oplus \mathbb{Z}\lambda_1\lambda_2 \cong H^*_{toric}(W, \mathbb{Z}).$
$Ann(-3\lambda_1 - 2\lambda_2) = \left\langle \lambda_1^2\lambda_2, 3\lambda_1^2 - 2\lambda_1\lambda_2 \right\rangle.$
W : generic bi-degree $(3, 2)$ K3 surface in $\mathbb{P}^2 \times \mathbb{P}^1$.

Singular loci of GKZ A-HGF

Discriminantal loci $D \subset \mathbb{C}^d$ of the family of varieties $Y_{\mathbf{s}} := \{x \in \mathbf{T}^{n-1}; f(x, 1, \mathbf{s}) = 0\}$.

$$s \in D \iff Y_{\mathbf{s}} : \text{singular}.$$

Amoeba $\mathrm{Log}(D)$: $\mathrm{Log}(D \cap (\mathbb{C}^*)^d)$ by
$\mathrm{Log}\colon (s_1, \dots, s_d) \mapsto (\log|s_1|, \dots, \log|s_d|)$.

Disjoint components $M_\rho, \rho \in [1, Q]$

$$\cup_{\rho=1}^{Q} M_\rho = \mathbb{R}^d \setminus \mathrm{Log}(D)$$

$Q :=$ number of vertices of the "secondary polytope" ($=$ the reduced defining equation of D) of A.

$$\mathcal{V}_\rho := \mathrm{Log}^{-1}(M_\rho) \subset \mathbb{C}^d \setminus D.$$

Proposition

(GKZ,Passare-Sadykov-Tsikh, Borisov-Horja)
$\forall M_\rho \subset \mathbb{R}^d \setminus \mathrm{Log}(D)$, $\rho \in [1;Q]$, $\exists \tilde{v}^{(\rho)} \in \mathbb{R}^d$ *such that*

$$C_\rho + \tilde{v}^{(\rho)} \subset M_\rho.$$

Convex hull (P_ρ) *in* $\mathbb{R}^d = -C_\rho^\vee$.

$C_\rho^\vee := \{w \in \mathbb{R}^d; < w, v > \geq 0, \forall v \in C_\rho\}$.
$\Phi_\gamma^{(\rho)}(\mathbf{s}) \in \mathcal{O}_{\mathcal{V}_\rho}$ *for all* γ *i.e.* $\forall \varphi_\gamma(\mathbf{z})$
$(\varphi_\gamma(\mathbf{z}+\mathbf{z}_0) = \varphi_\gamma(\mathbf{z}), \forall \mathbf{z}_0 \in \mathbb{Z}^d)$ *given by*

$$\varphi_\gamma(\mathbf{z}) = \prod_{p \in I_\gamma} \left(\frac{\sin \pi < \mathfrak{b}_p, \mathbf{z} >}{\pi e^{\pi i < \mathfrak{b}_p, \mathbf{z} >}} \right) \tilde{\varphi}_\gamma(\mathbf{z})$$

for $I_\gamma \subset [1;N]$.

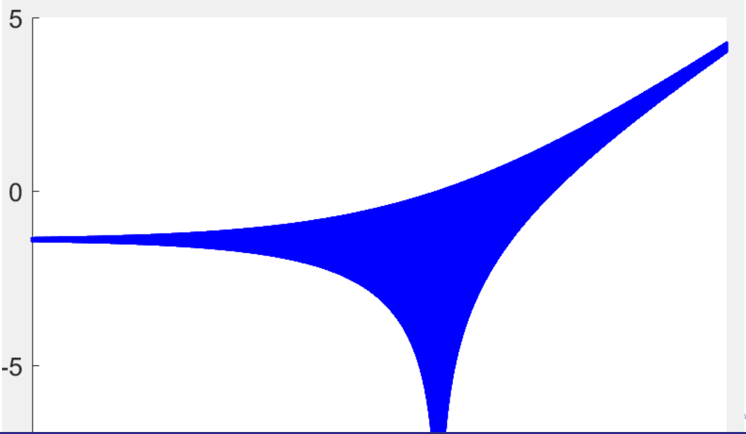
$$\Phi_\gamma^{(\rho)}(\mathbf{s}) = \sum_{\mathbf{z}\in P_\rho} Res_{\mathbf{z}} \frac{\bar{\phi}_\gamma(\mathbf{z})\mathbf{s}^{\mathbf{z}}}{\prod_{1\leq p\leq N}\Gamma(1+<\mathfrak{b}_p,\mathbf{z}>)}, \quad (6.1)$$

Suffix $\rho \in [1;Q]$
$\leftrightarrow$ regular triangulation $\mathcal{T}_\rho$
$\leftrightarrow$ vertex of the secondary polytope of A
$\leftrightarrow$ P_ρ : support of the power series $\leftrightarrow$ *cone* $-C_\rho^\vee$
$\leftrightarrow$ $\mathcal{V}_\rho$ domain of convergence $\leftrightarrow$ *cone* C_ρ .
Here P_ρ :

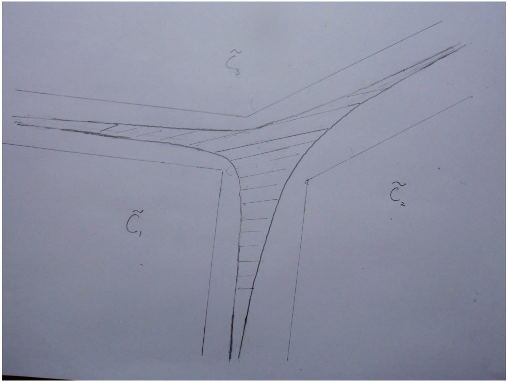
$$-<\mathfrak{b}_p,\mathbf{z}>\in \mathbb{Z}_{\leq 0} \text{ for } p\in \mathcal{J}_\rho \subset [1;N] \qquad (6.2)$$

where $\mathcal{J}_\rho \subset [1;N]$, $|\mathcal{J}_\rho| = rank(\mathfrak{b}_p)_{p\in\mathcal{J}_\rho} = d$.

Amoeba of $s_1 = (\frac{z}{3z+2})^3, s_2 = (\frac{1}{3z+2})^2$: D for Example $n = 4$.

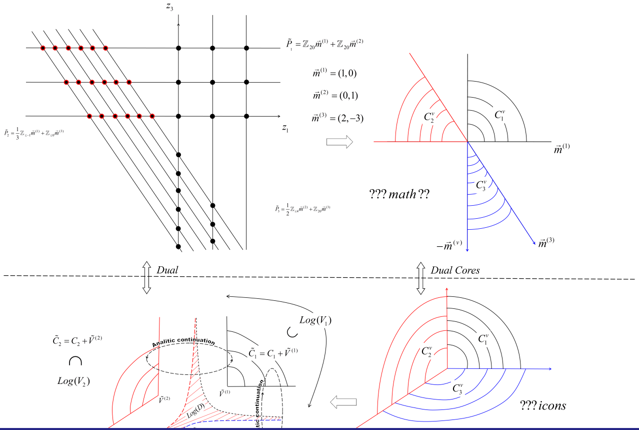


Amoeba (D for Example $n = 4$.) and its recession cones

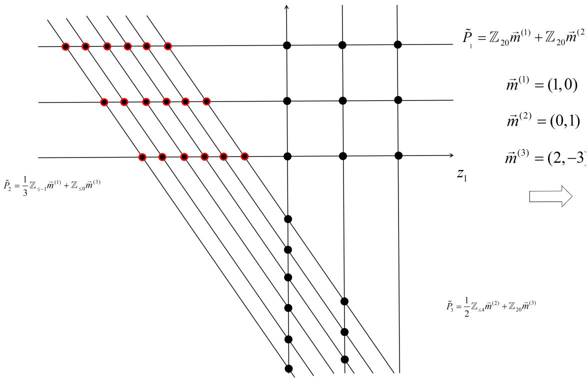


$$\tilde{C}_\rho = C_\rho + \tilde{v}^{(\rho)}$$

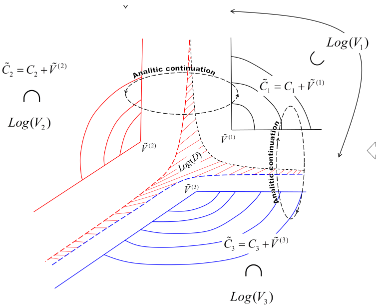
Cones associated to secondary fan



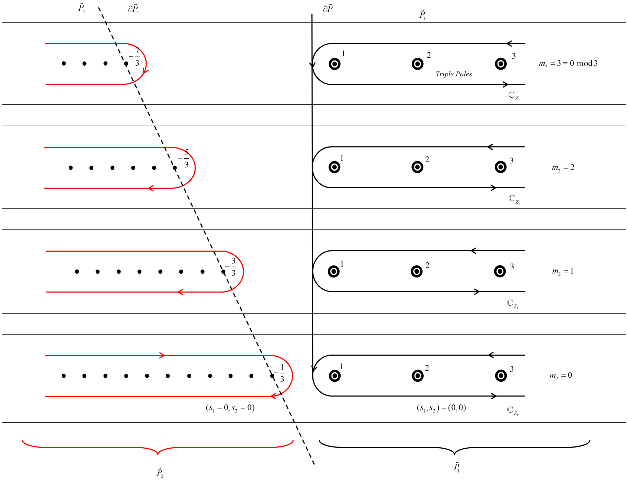
Supports P_ρ of the HG series $\Phi_\gamma^{(\rho)}(\mathbf{s})$.



Amoebas of the discriminantal loci



Example, n=4. Contour throw



GKZ A-HG Series

Λ : Gorenstein cone.

Consider a $\mathcal{R}_{A,\mathcal{T},\mathbb{C}} = \mathcal{R}_{A,\mathcal{T}} \otimes \mathbb{C}$ (or $\bar{\mathcal{R}}_{A,\mathcal{T},\mathbb{C}} := \bar{\mathcal{R}}_{A} \otimes \mathbb{C}$) valued solution to GKZ (or equivalently to Horn HG system)

$$\Phi_1(\mathbf{s}, \boldsymbol{\lambda}) := \sum_{\mathbf{m}\in P_1} \varpi(\mathbf{m}+\boldsymbol{\lambda})\mathbf{s}^{\mathbf{m}+\boldsymbol{\lambda}} \quad in\ \mathcal{R}_{A,\mathcal{T}} \otimes \mathcal{O}_{\mathcal{V}_1} \quad (6.3)$$

with

$$\varpi(z) = \frac{1}{\prod_{p=1}^{N} \Gamma(< \mathfrak{b}_p, z > +1)}. \quad (6.4)$$

Here summation runs over $\mathbf{m} = (m_1, \mathbf{m}')$: solutions to d linearly independent linear equations

$m_1 \in \mathbb{Z}_{\geq 0}, < \mathfrak{b}_p, (m_1, \mathbf{m}') > \in \mathbb{Z}_{\geq 0}$

for $p \in J_1 \subset I_+^{(1)} \cup I_0^{(1)}$, $|J_1| = d$.

Monodromy theorem

Theorem

(P.R.Horja, 1999) $\bar{Y}_s$: *Calabi-Yau hypersurface defined by a reflexive polytope* $\Delta(F)$.
$\Lambda = \sum_{p=1} \mathbb{R}_{\geq 0}\bar{\alpha}_p$: *Gorenstein cone.*
The monodromy of $\Phi_1(\mathbf{s})$ *along a loop* $\mathcal{V}_1 \rightarrow \mathcal{V}_2 \rightarrow \mathcal{V}_1$*;*

$$\Phi_1(\mathbf{s}) \rightarrow \Phi_1(\mathbf{s}) - 2\pi i \sum_{\mathbf{m}' \in \mathbb{L}'_1} \prod_{p \in I^{(1)}_-} (1 - e(< \mathfrak{b}_p, \boldsymbol{\lambda} >))$$

$$Res^+_{\zeta_1}\left(\frac{\varpi(\zeta_1, \mathbf{m}' + \boldsymbol{\lambda}')s_1^{\zeta_1}\mathbf{s}'^{\mathbf{m}'}}{\prod_{q \in I^{(1)}_+} 1 - e(- < \mathfrak{b}_q, (\zeta_1, \boldsymbol{\lambda}') >)} \right).$$

(*Picard-Lefschetz type pseudo-reflection*)

(Not necessarily Gorenstein) Delsarte hypersurface. S.T. 2022.

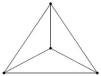
Consider affine var. $Y_s = \{x \in {\mathbb{C}^*}^n; f_0(x) + s = 0$ with $s \in \mathbb{C}$,

$$f_0(x) = (\sum_{j=1}^{n} x^{\alpha(j)} + 1)x^{-\alpha(n+2)}. \tag{7.1}$$

$\{0\} \in \Delta(f_0)^{int}$: a $n-$dim. simplex

$$\gamma = n! vol(\Delta(f_0)) > 0. \tag{7.2}$$

$\mathbf{B} = (B_1, \cdots, B_{n+1})$, B_q : volume of a subdiv. simplex, $q \in [1, n+1]$, $g.c.d.\mathbf{B} = 1$. $\sum_{q=1}^{n+1} B_q = \gamma$.



Stanley-Reisner ring defined for the unimodular triangulation $\mathcal{T}$ of $\Delta(f_0)$.

$$\mathcal{R}_{A,\mathcal{T}} \cong \mathbb{C}[\lambda]/ < \lambda^{\gamma} > \cong H^*(\mathbb{P}_{B}, \mathbb{C}).$$

$\mathbb{P}_{\mathbf{B}}$ Fano var. $\longleftrightarrow \frac{\gamma}{B_j} \in \mathbb{Z}, \forall j \in [1; n+1]$.
By a Berglund -Hübsch type transposition of f_0 we construct $f^T(y)$ with weight $\gamma = |\mathbf{B}|$, (W : C.Y. if smooth)

$$W = \{y \in \mathbb{P}_{\mathbf{B}}; f^T(y) = 0\}.$$

$$\bar{\mathcal{R}}^{\mathbb{C}}_{A,\mathcal{T}} \cong \mathbb{C}[\lambda]/ < \lambda^{\bar{\gamma}} > \cong H^*(W, \mathbb{C}). \qquad (7.3)$$

$$\bar{\gamma} = \sharp\{\text{poles of } \frac{\Gamma(-\gamma z)}{\prod_{q=1}^{n+1}\Gamma(-B_q z)}, z \in [0,1)\}.$$

Consider the H.G. series arising from the period integral of Y_s

$$\Psi(s,\lambda) = \sum_{m\geq 0} \frac{\Gamma(\gamma(m+\lambda)+1)}{\prod_{q=1}^{n+1}\Gamma(B_q(m+\lambda)+1)}(-s)^{-\gamma(m+\lambda)} =$$

$$= \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0-i\infty} \frac{\Gamma(\gamma(z+\lambda)+1)e^{-\pi i z}}{\prod_{q=1}^{n+1}\Gamma(B_q(z+\lambda)+1)}\Gamma(z)\Gamma(1-z)(-s)^{-\gamma(z+\lambda)}$$

$$= Res_{z\in\mathbb{Z}_{\geq 0}} \frac{1}{e(z)-1}\frac{\Gamma(\gamma(z+\lambda)+1)}{\prod_{q=1}^{n+1}\Gamma(B_q(z+\lambda)+1)}(-s)^{-\gamma(z+\lambda)}dz.$$

$$e(z) = e^{2\pi i z}, \lambda \in H^*(\mathbb{P}_{\mathrm{B}},\mathbb{C}).$$

Monodromy h_1 around $s = \frac{\gamma}{(\prod_{q=1}^{n+1} B_q)^{\frac{1}{\gamma}}} \in \mathbb{R}$ of p.i. of $\bar{Y}_s$,

$h_1 : \Psi(s, \lambda)$

$$\longrightarrow \Psi(s,\lambda) - 2\pi\mathbf{i}Res_{z=0}\frac{(1-e(-\gamma z))}{\prod_{q=1}^{n+1}(1-e(-B_q z))}\Psi(s,z),$$

(Pseudo-reflection). $\lambda \in H^*(W)$.
Todd class of W

$$Todd_W = \frac{1-e^{-\gamma[D]}}{\gamma[D]}(\prod_{q=1}^{n+1}\frac{B_q[D]}{1-e^{-B_q[D]}}) \ \ mod([D]^{\bar{\gamma}}) \ in \ H^*(W)$$

with $[D] \in H^2(W)$. The above residue is equivalent to the $[D]^{n-1}$ part of

$$Todd_W \cdot \Psi(s, [D]/2\pi i) \ in \ H^*(W) \otimes \mathcal{O}.$$

(Kontsevich conjecture type result)

Example: d=2. P.R. Horja 1999

Consider a $(n+1) \times (n+4)$ matrix for smooth C.Y. $W \subset \tilde{\mathbb{P}}(2q_1, \cdots, 2q_n, 1, 1)$ a blow up obtained torically by adding a vector to the defining fan of $\mathbb{P}(2q_1, \cdots, 2q_n, 1, 1)$ (For $n = 2$)

$$\mathrm{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & -2q_1 & -q_1 \\ 0 & 0 & 1 & 0 & -2q_2 & -q_2 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathrm{B} = \begin{pmatrix} -q & q_1 & q_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{pmatrix},$$

with $q = 1 + \sum_{j=1}^{n} q_j$.

$$\mathcal{R}_{A,\mathbb{C}} \cong \mathbb{C}[\boldsymbol{\lambda}]/ < \lambda_1^n(\lambda_1 - 2\lambda_2), \lambda_2^2 > \cong H^*(\tilde{\mathbb{P}}(2q_1, \cdots, 2q_n, 1, 1), \mathbb{C}).$$

$$\bar{\mathcal{R}}_{A,\mathbb{C}} \cong \mathcal{R}_{A,\mathbb{C}}/Ann(-q\lambda_1)$$

$$\cong \mathbb{C}[\lambda]/ < \lambda_1^{n-1}(\lambda_1 - 2\lambda_2), \lambda_2^2 > \cong H^*(W, \mathbb{C}).$$

$\Psi_1(\mathbf{s}, \boldsymbol{\lambda}) \in$ sol (A-GKZ HGS): periods of $\bar{Y}_s$
$\in \bar{\mathcal{R}}_{A,\mathbb{C}} \otimes \mathcal{O}_{\mathcal{V}_1}$.

$$\Psi_1(\mathbf{s}, \boldsymbol{\lambda}) = \sum_{\mathbf{m} \in P_1 = (\mathbb{Z}_{\geq} 0)^2} \frac{\Gamma(q(m_1+\lambda_1))s^{\mathbf{m}+\boldsymbol{\lambda}}}{\prod_{j=1}^n \Gamma(q_j(m_1+\lambda_1)+1)\Gamma(m_1+\lambda_1-2(m_2+\lambda_2)+1)\Gamma(m_2+\lambda_2+1)^2}.$$

with $q = 1 + \sum_{j=1}^n q_j$.

Discriminantal divisor Δ_0 :

$$\Delta_0 = \{\mathbf{s}; s_2 = \frac{1}{4}(1 - \frac{\prod_{j=1}^{n} q_j^{q_j}}{q^q}\frac{1}{s_1})^2\}.$$

Singular loci

$$\Delta_0 \cup \{s_1 = 0\} \cup \{\frac{1}{s_1} = 0\} \cup \{s_2 = 0\} \cup \{s_2 = 1/4\}.$$

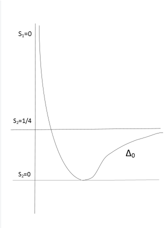
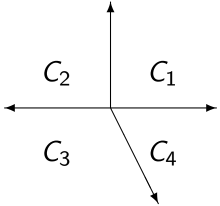
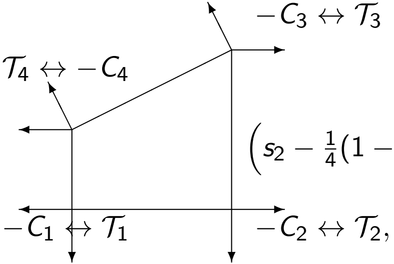


Figure: Singular loci

Secondary polytope, secondary fan





$$\left(s_2 - \frac{1}{4}(1 - const.\frac{1}{s_1})^2\right)(s_2 - 1/4)$$

Monodromy around the discriminantal divisor Δ_0 is given by

$$\Psi_1(\mathbf{s},\boldsymbol{\lambda}) \to \Psi_1(\mathbf{s},\boldsymbol{\lambda}) - 2\pi\mathbf{i}Res_{\mathbf{z}=\mathbf{0}}T(\mathbf{z})\Psi_1(\mathbf{s},\mathbf{z})$$

$$T(\mathbf{z}) = \frac{1-e(-qz_1)}{(1-e(-z_1+2z_2))(1-e(-z_2))^2\prod_{j=1}^{n}(1-e(-q_jz_1))}$$

$$2\pi\mathbf{i}Res_{\mathbf{z}=\mathbf{0}}T(\mathbf{z})\Psi_1(\mathbf{s},\mathbf{z}) = \int_W Todd_W([\mathbf{D}])\Psi_1(s,[\mathbf{D}]/2\pi i).$$

$$Todd_W([\mathbf{D}]) = Todd_W([D_1],[D_2]) =$$

$$= \frac{1-e^{-q[D_1]}}{q[D_1]}\frac{[D_1]-2[D_2]}{1-e^{-[D_1]+2[D_2]}}\left(\frac{[D_2]}{1-e^{-[D_2]}}\right)^2\prod_{j=1}^{n}\frac{q_j[D_1]}{1-e^{-q_j[D_1]}}$$

with $[D_1],[D_2]\in H^2_{toric}(W)$. (Kontsevich conjecture type result).