Stokes structure of difference modules

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Plan

- Main result
    - Riemann-Hilbert correspondence for mild difference modules
- Motivations/Expected application
    - Mellin transformation

Main result

Riemann-Hilbert correspondence for mild difference modules

- Main result
- Motivations/Expected App.

Introduction: Rough Sketch of the main theorem

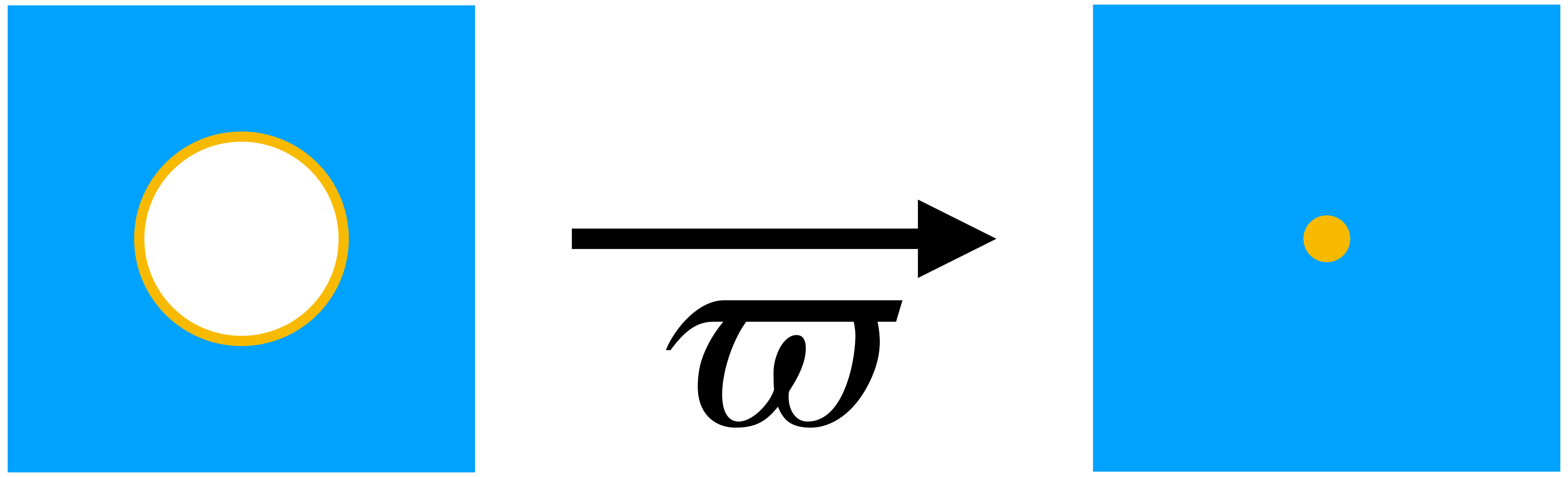
Riemann-Hilbert correspondence (Deligne-Malgrange)

$$\mathrm{RH} : \mathsf{Mer}(\mathbb{C},0) \xrightarrow{\sim} \mathsf{Stokes}(S^1), \quad (M,\nabla) \mapsto (\mathscr{H}^0\widetilde{\mathrm{DR}}(M), \mathscr{H}^0\mathrm{DR}_{\leqslant\bullet}(M))$$

- Germ of merom. conn.
- Stokes filtered loc. sys. on S^1
- cohomology of de Rham complex

Real oriented blow up

$$\widetilde{\mathbb{C}} := \{(z, e^{i\theta}) \in \mathbb{C} \times S^1 \mid z = |z| e^{i\theta}\}$$
$$\xrightarrow{\varpi} \mathbb{C}, \quad \varpi(z, e^{i\theta}) = z$$



Main theorem gives a mild difference analog of this theorem.

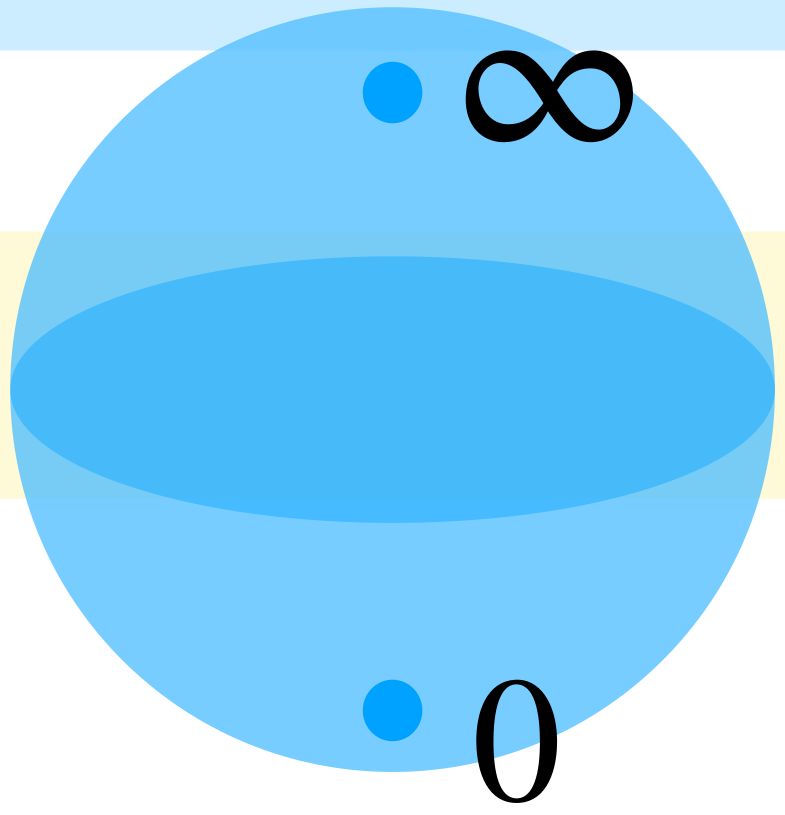
Difference modules

- $K = \mathbb{C}\{t\}[t^{-1}]$: Field of convergent Laurent series.
- $\phi\colon K \to K$: Automorphism of fields defined as $\phi(f)(t) = f\left(\frac{t}{1+t}\right), \quad f \in K.$

Definition A difference module (over (K, ϕ)) is a pair $\mathscr{M} = (\mathscr{M}, \psi)$ of

- a finite dimensional K-vector space $\mathscr{M}$, and
- an automorphism $\psi\colon \mathscr{M} \to \mathscr{M}$ such that $\psi(fv) = \phi(f)\psi(v)$ for $f \in K, v \in \mathscr{M}$.

Remark Set $s = t^{-1}$. Then we have $\phi(f)(s) = f(s+1), \quad f \in K.$



Examples of difference modules

Regular singular modules Take a matrix $G \in \mathrm{End}(\mathbb{C}^r)$ and set

$$\mathscr{R}_G := (K^{\oplus r}, \psi_G), \quad \psi_G := (1+t)^G \phi^{\oplus r}.$$

t^G

A difference module formally isomorphic to $\mathscr{R}_G$ is called regular singular.

$s = t^{-1}$

Exponential modules Take $d \in \mathbb{Z}, c \in \mathbb{C}$ and set $\mathfrak{a}(s) = ds \log s + cs$ and

$$\mathscr{E}^{\mathfrak{a}} := (K, \psi_{\mathfrak{a}}), \quad \psi_{\mathfrak{a}} := \exp(\mathfrak{a}(s+1) - \mathfrak{a}(s))\phi.$$

$\psi(e^{-a(s)})$

Remark We have $\exp(\mathfrak{a}(s+1) - \mathfrak{a}(s)) \in K = \mathbb{C}(\{t\})$.

$= e^{a(s+1)-a(s)} \cdot e^{-a(s+1)}$

$= e^{-a(s)}$

Mild difference modules

- $\widehat{K}_m := \mathbb{C}((t^{1/m}))$: Field of formal Laurent series with $\widehat{\phi}_m(t^{1/m}) = t^{1/m}(1+t)^{-1/m}$.

Formal decomposition theorem ∀difference module $\mathscr{M}$, $\exists m \in \mathbb{Z}_{>0}$ such that

$$\mathscr{M} \otimes_K \widehat{K}_m \simeq \bigoplus_{k=1}^{\ell} \widehat{\mathscr{E}}^{\mathfrak{a}_k} \otimes_K \mathscr{R}_{G_k}$$

where $G_k \in \mathrm{End}(\mathbb{C}^{r_k})$, $\mathfrak{a}_k = d_k s \log s + \sum_{j=1}^{m} c_{j,k}\, s^{\frac{j}{m}}$ $\quad (d_k \in m^{-1}\mathbb{Z}, c_{j,k} \in \mathbb{C})$, and

$$\widehat{\mathscr{E}}^{\mathfrak{a}_k} = (\widehat{K}_m, \widehat{\psi}_{\mathfrak{a}_k}), \quad \widehat{\psi}_{\mathfrak{a}_k} = \exp(\mathfrak{a}_k(s+1) - \mathfrak{a}_k(s))\,\widehat{\phi}_m.$$

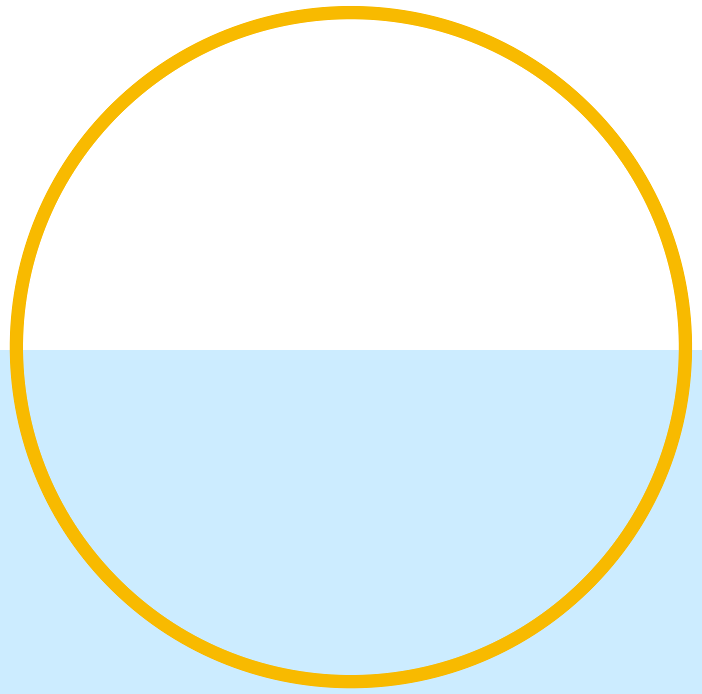
Definition $\mathscr{M}$ is called mild if we have $d_k = 0$ for every $k = 1,\ldots,\ell$.

wild $\quad d_k \neq 0 \quad \exists \quad k.$

Stokes filtered locally free sheaves 1

Sheaf of rings on a circle

For $a < b$, set $(a, b) := \{e^{i\theta} \in S^1 \mid a < \theta < b\}$.



Definition For a connected open subset $U \subset S^1$, we set

$$\mathscr{A}_{\mathrm{per}}^{\leqslant 0}(U) = \begin{cases} \mathbb{C}\{u^{-1}\} & (U \subset (0, \pi)) \\ \mathbb{C}\{u\} & (U \subset (-\pi, 0)) \\ \mathbb{C} & (U \cap \{\pm 1\} \neq \varnothing) \end{cases}$$

, which defines a sheaf of rings on S^1.

We then set $\mathscr{A}_{\mathrm{per}} := \sum_{n \in \mathbb{Z}} u^n \mathscr{A}_{\mathrm{per}}^{\leqslant 0} \subset \mathbb{C}[\![u, u^{-1}]\!]_{S^1}$, which is also a sheaf of rings on S^1.

Remark We will regard u as $\exp(2\pi i s)$.

Stokes filtered locally free sheaves 2
Sheaf of ordered set of indexes

$\tilde{\imath}: S^1 \hookrightarrow \widetilde{\mathbb{C}} \hookleftarrow \mathbb{C}^*: \tilde{\jmath}$: natural inclusions. $\mathcal{O}_{\mathbb{C}^*}$: sheaf of holomorphic functions.

Definition For a connected open subset $U \subset S^1$, we set

$$\mathcal{I}(U) = \left\{ \mathfrak{a} \in \tilde{\imath}^{-1}\tilde{\jmath}_*\mathcal{O}_{\mathbb{C}^*}(U) \;\middle|\; \mathfrak{a}(s) = \sum_{j=1}^{m} c_j s^{\frac{j}{m}},\ c_j \in \mathbb{C}, m \in \mathbb{Z}_{>0} \right\}$$

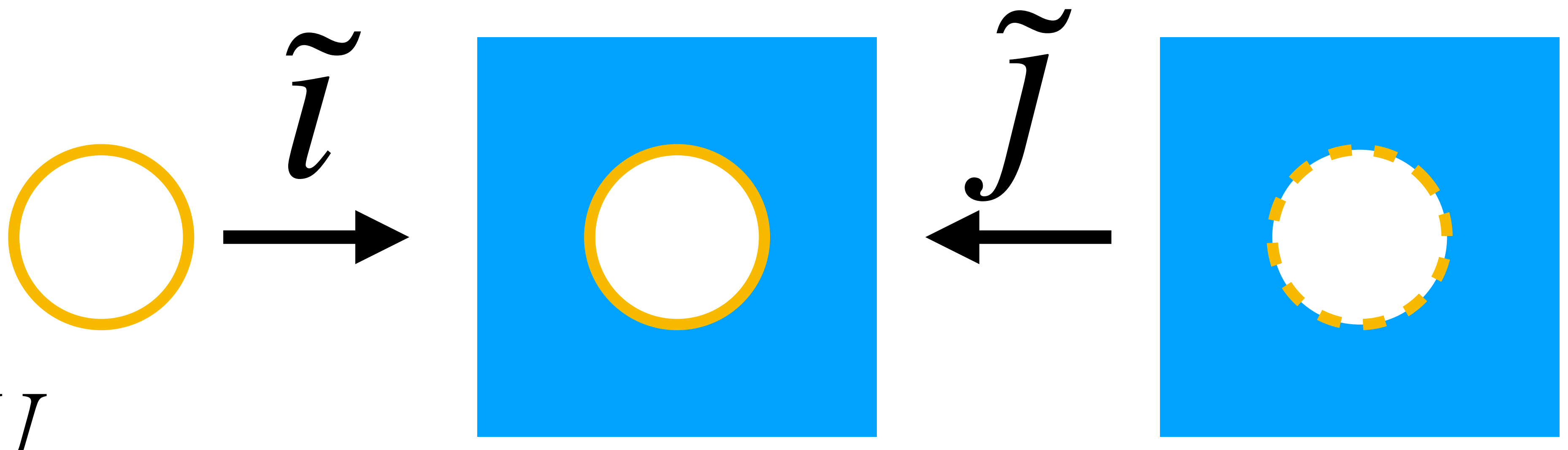
$|e^{\mathfrak{a}(s)}| = e^{\mathrm{Re}(\mathfrak{a}(s))}$

where we fix a branch of $\log s$ and hence $s^{\frac{1}{m}} = \exp(m^{-1}\log s)$.

We define the order $<_U$ ($\leqslant_U$) on $\mathcal{I}(U)$ by

$\mathfrak{a} <_U \mathfrak{b}$ $(\mathfrak{a} \leqslant_U \mathfrak{b}) \Leftrightarrow (\mathfrak{a} = \mathfrak{b}$ or)

$\mathrm{Re}[\mathfrak{a}(s)] < \mathrm{Re}[\mathfrak{b}(s)]$ for $|s| \gg 0, -\arg(s) \in U$.



$\tilde{\imath}: S^1 \hookrightarrow \widetilde{\mathbb{C}} \hookleftarrow \mathbb{C}^*: \tilde{\jmath}$

Stokes filtered locally free sheaves 3
Stokes filtrations

Definition Let $\mathscr{L}$ be an $\mathscr{A}_{\mathrm{per}}$-module. A pre-Stokes filtration on $\mathscr{L}$ is a family $\{\mathscr{L}_{\leqslant\mathfrak{a}} \subset \mathscr{L}_{|U} \mid U \subset S^1, \mathfrak{a} \in \mathscr{I}(U)\}$ of $\mathscr{A}_{\mathrm{per}}^{\leqslant 0}$-submodules s.t.

- If $\mathfrak{a}_{|V} = \mathfrak{b}$ for $V \subset U$, $\mathfrak{a} \in \mathscr{I}(U)$, and $\mathfrak{b} \in \mathscr{I}$, then $\mathscr{L}_{\leqslant\mathfrak{a}|U} = \mathscr{L}_{\leqslant\mathfrak{b}}$.
- If $\mathfrak{a} \leqslant_U \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathscr{I}(U)$, then $\mathscr{L}_{\leqslant\mathfrak{a}} \subset \mathscr{L}_{\leqslant\mathfrak{b}}$.
- For $n \in \mathbb{Z}$ and $\mathfrak{a} \in \mathscr{I}(U)$, we have the equality $u^n \mathscr{L}_{\leqslant\mathfrak{a}} = \mathscr{L}_{\leqslant\mathfrak{a}+2\pi ins}$.

$u = \exp(2\pi i s)$

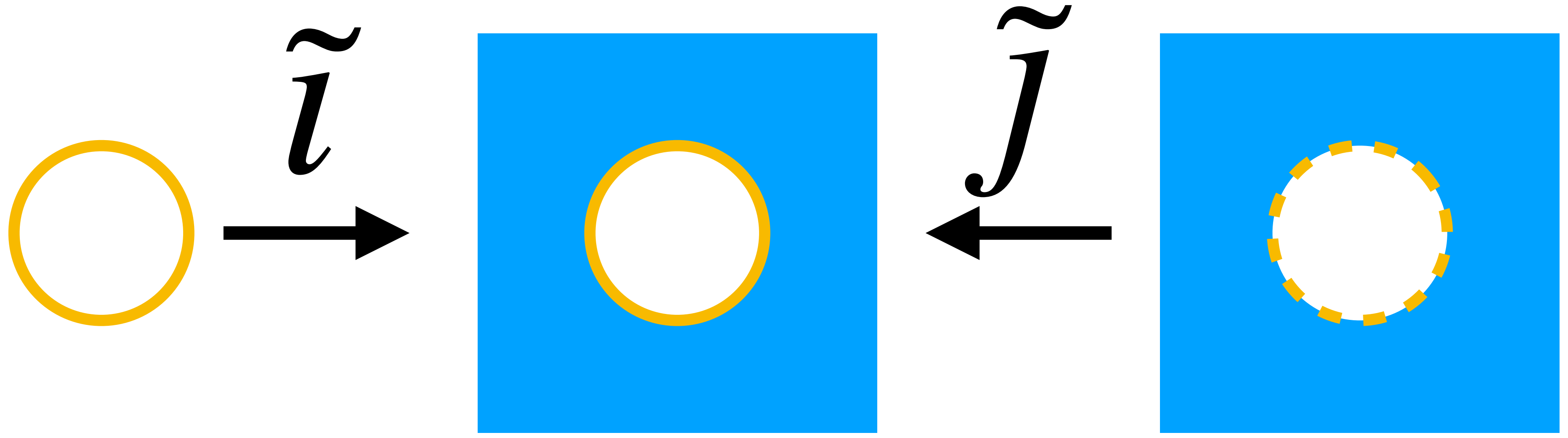
$$\Rightarrow \quad \mathscr{L}_{<\mathfrak{a}} = \sum_{\mathfrak{b} <_U \mathfrak{a}} \mathscr{L}_{\leqslant\mathfrak{b}}, \quad \mathrm{gr}_{\mathfrak{a}}(\mathscr{L}) = \mathscr{L}_{\leqslant\mathfrak{a}} / \mathscr{L}_{<\mathfrak{a}}, \quad \mathrm{gr}(\mathscr{L})_{|U} = \bigoplus_{\mathfrak{a} \in \mathscr{I}(U)} \mathrm{gr}_{\mathfrak{a}}(\mathscr{L}).$$

A pre-Stokes filtration on a locally free $\mathscr{A}_{\mathrm{per}}$-module is called a Stokes filtration if

$\forall x \in S^1, \exists U \subset S^1$ s.t. $\exists \eta : \mathrm{gr}(\mathscr{L}) \otimes_{\mathbb{C}[u,u^{-1}]} \mathscr{A}_{\mathrm{per}|U} \xrightarrow{\sim} \mathscr{L}_{|U}$ with $\mathrm{gr}(\eta) = \mathrm{id}$.

De Rham functors

- $\widetilde{\mathscr{O}} = \tilde{\imath}^{-1}\tilde{\jmath}_*\mathscr{O}_{\mathbb{C}^*}$, $\widetilde{\phi}(f)(t) = f(t(1+t)^{-1})$.
- $\mathscr{A}^{\leqslant 0} \subset \widetilde{\mathscr{O}}$: moderate growth functions.
- $\mathscr{A}^{<0} \subset \widetilde{\mathscr{O}}$: rapid decay functions.



$\tilde{\imath}: S^1 \hookrightarrow \widetilde{\mathbb{C}} \hookleftarrow \mathbb{C}^*: \tilde{\jmath}$

Definition For a mild difference module $\mathscr{M} = (\mathscr{M}, \psi)$, we define complexes

degree 0 and 1

- $\widetilde{\mathrm{DR}}(\mathscr{M}) = [\widetilde{\mathscr{O}} \otimes \mathscr{M}_{S^1} \xrightarrow{\widetilde{\psi}-\mathrm{id}} \widetilde{\mathscr{O}} \otimes \mathscr{M}_{S^1}]$,
- $\mathrm{DR}_{\leqslant 0}(\mathscr{M}) = [\mathscr{A}^{\leqslant 0} \otimes \mathscr{M}_{S^1} \xrightarrow{\widetilde{\psi}-\mathrm{id}} \mathscr{A}^{\leqslant 0} \otimes \mathscr{M}_{S^1}]$, and $\mathrm{DR}_{<0}(\mathscr{M}) = [\mathscr{A}^{<0} \otimes \mathscr{M}_{S^1} \xrightarrow{\widetilde{\psi}-\mathrm{id}} \mathscr{A}^{<0} \otimes \mathscr{M}_{S^1}]$.

Theorem (S) If $\mathscr{M}$ is mild, $\mathscr{H}^i(\mathrm{DR}_{\leqslant 0}(\mathscr{M})) = \mathscr{H}^i(\mathrm{DR}_{<0}(\mathscr{M})) = 0 \quad (i \neq 0)$.

Main theorem

For $\mathfrak{a} \in \mathscr{I}(U), U \subset S^1$: open, we set $\mathrm{DR}_{\leqslant \mathfrak{a}}(\mathscr{M}) = [e^{\mathfrak{a}}\mathscr{A}^{\leqslant 0}_{|U} \otimes \mathscr{M}_U \xrightarrow{\widetilde{\psi}-\mathrm{id}} e^{\mathfrak{a}}\mathscr{A}^{\leqslant 0}_{|U} \otimes \mathscr{M}_U]$.

There exists a unique $\mathscr{A}_{\mathrm{per}}$-submodule $\mathrm{Per}(\mathscr{M}) \subset \mathscr{H}^0\widetilde{\mathrm{DR}}(\mathscr{M})$ such that

$$\mathrm{Per}(\mathscr{M})_{|U} = \sum_{\mathfrak{a} \in \mathscr{I}(U)} \mathscr{H}^0\mathrm{DR}_{\leqslant \mathfrak{a}}(\mathscr{M}) \,. \subsetneq \mathscr{H}^0(\widetilde{\mathrm{DR}}(\mathscr{M}))$$

Theorem (S. arXiv: 2212.10753) Let $\mathscr{M}$ be a mild difference module.

- The pair $\mathrm{RH}(\mathscr{M}) := (\mathrm{Per}(\mathscr{M}), \mathscr{H}^0\mathrm{DR}_{\leqslant \bullet}(\mathscr{M}))$ is a Stokes filtered $\mathscr{A}_{\mathrm{per}}$-module.
- The correspondence $\mathrm{RH} : \mathsf{Diffc}^{\mathrm{mild}} \to \mathsf{St}(\mathscr{A}_{\mathrm{per}})$ is an equivalence of categories.

cat. of mild difference mod.

cat. of Stokes filtered $\mathscr{A}_{\mathrm{per}}$-mod.

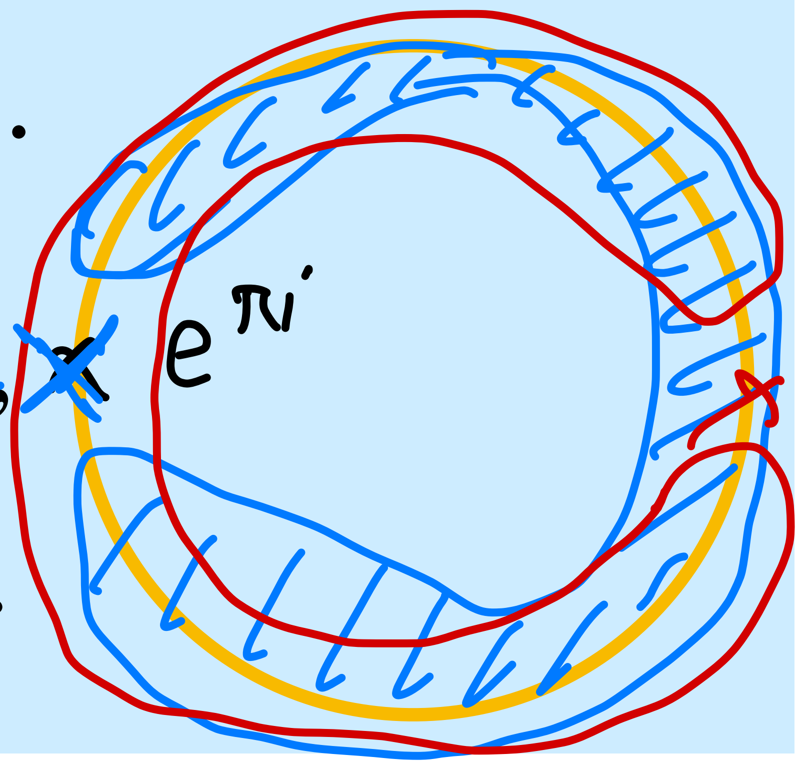
Rank one examples and Gamma functions

Theorem (S) $\mathrm{Per}(K, \phi) = \mathscr{A}_{\mathrm{per}}$, where $K = \mathscr{O}_t(*0) = \mathbb{C}\{t\}[t^{-1}]$.

Remark Concerning the 'wild Stokes filtration' with '$\leqslant_U s \log s$' cause a problem.

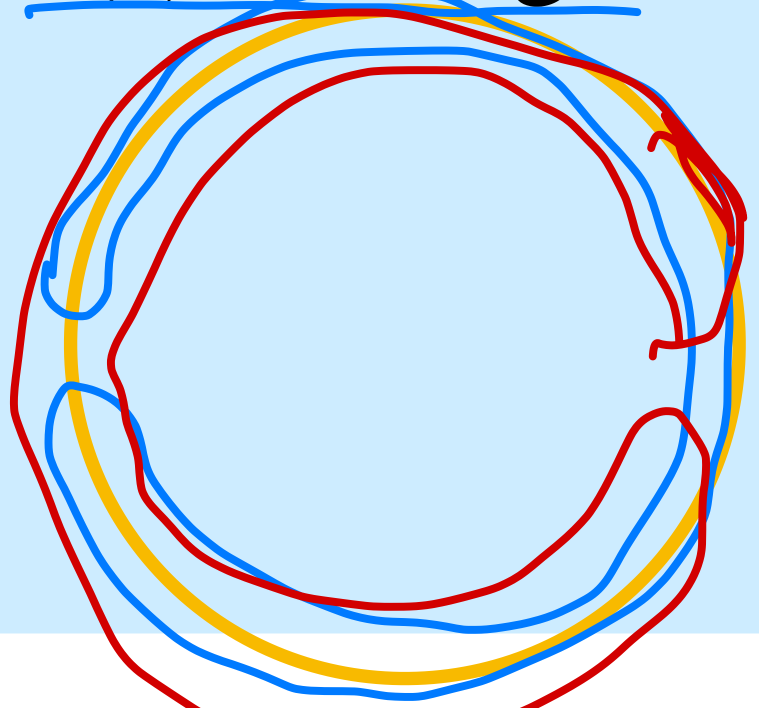
Regular singular modules Take $\alpha \in \mathbb{C} \backslash \mathbb{Z}$ and set $\mathscr{B}_\alpha := (K, (1 + \alpha t)\phi)$.

$$\mathrm{Per}(\mathscr{B}_\alpha)_{|U} = \begin{cases} \mathscr{A}_{\mathrm{per}|U} \Gamma(s)/\Gamma(s+\alpha) & (e^{\pi i} \notin U), \\ \mathscr{A}_{\mathrm{per}|U}(1-u)\Gamma(s)/(1-e^{2\pi i \alpha}u)\Gamma(s+\alpha) & (e^{0} \notin U). \end{cases}$$



Twisted Gamma module Set $\mathscr{E}_\Gamma = (K, \psi_\Gamma)$, $\psi_\Gamma = \exp(\mathfrak{l}(s+1) - \mathfrak{l}(s))t\phi$, $\mathfrak{l}(s) = s \log s$.

$$\mathrm{Per}(\mathscr{E}_\Gamma)_{|U} = \begin{cases} \mathscr{A}_{\mathrm{per}|U} s^{-s}\Gamma(s) & (e^{i\pi} \notin U), \\ \mathscr{A}_{\mathrm{per}|U}(1-u)s^{-s}\Gamma(s) & (e^{0} \notin U). \end{cases}$$



Riemann-Hilbert correspondence in wild case

Replacing $\mathscr{I}$ with $\mathscr{I}^{\mathrm{wild}}$,
We define wild version of Stokes structure

Definition For a connected open subset $U \subset S^1$, we set

$$\mathscr{I}^{\mathrm{wild}}(U) = \left\{ \mathfrak{a} \in \tilde{\imath}^{-1}\tilde{\jmath}_*\mathscr{O}_{\mathbb{C}^*}(U) \,\middle|\, \mathfrak{a}(s) = \frac{\ell}{m} s \log s + \sum_{j=1}^{m} c_j s^{\frac{j}{m}}, c_j \in \mathbb{C}, m \in \mathbb{Z}_{>0}, \ell \in \mathbb{Z} \right\}$$

where we fix a branch of $\log s$ and hence $s^{\frac{1}{m}} = \exp(m^{-1} \log s)$.

Conjecture There exists an equivalence of categories:

$$\mathrm{RH} : \mathrm{Diffc} \xrightarrow{\sim} \mathrm{St}^{\mathrm{wild}}(\mathscr{A}_{\mathrm{per}})$$

cat. of any difference mod.

cat. of wild Stokes filtered $\mathscr{A}_{\mathrm{per}}$-mod.

Motivation/Expected applications

Mellin transformations

- Main result
- Motivations/Expected applications
    - Mellin transformation

Algebraic Mellin transformation

Isomorphism of rings:

$$\mathscr{D}_{\mathbb{G}_m} = \mathbb{C}[x, x^{-1}]\langle x\partial_x \rangle \simeq \mathfrak{M} = \mathbb{C}[s]\langle \phi, \phi^{-1} \rangle$$

$$x \leftrightarrows \phi$$

$$x\partial_x \leftrightarrows -s$$

For $\mathscr{D}_{\mathbb{G}_m}$-module $\mathscr{N}$, we set

$$\mathfrak{M}(\mathscr{N}) := \mathfrak{M} \otimes_{\mathscr{D}} \mathscr{N}$$

Theorem(López, arXiv:1804.09776v3) For a holonomic $\mathscr{D}_{\mathbb{G}_m}$-module $\mathscr{N}$, we have

$$\mathfrak{M}(\mathscr{N}) \otimes_{\mathbb{C}[s]} \mathbb{C}((s^{-1})) \simeq \bigoplus_{\star \in \mathrm{Sing}(\mathscr{N}) \cup \{0,\infty\}} \mathfrak{M}_{\star,\infty}(\mathscr{N}).$$

local Mellin trans.

- For a regular holonomic $\mathscr{D}_{\mathbb{G}_m}$-module $\mathscr{N}$, $\mathfrak{M}(\mathscr{N})_\infty := K \otimes_{\mathbb{C}[s]} \mathfrak{M}(\mathscr{N})$ is mild.

Question Can we describe $\mathrm{RH}(\mathfrak{M}(\mathscr{N})_\infty)$ in terms of $\mathrm{DR}(\mathscr{N})$?

perverse sheaf

← Stokes filtration

Stokes structure of Mellin transformations

Question Can we describe $\mathrm{RH}(\mathfrak{M}(\mathcal{N})_\infty)$ in terms of $\mathrm{DR}(\mathcal{N})$ or $\mathrm{Sol}(\mathcal{N})$?

Assume $\mathcal{N} = (E, \nabla)$ is an algebraic connection on $U = \mathbb{G}_m \backslash S$, $S = \{s_1, \ldots, s_\ell\}$.

Integral presentation of solutions of $\mathfrak{M}(\mathcal{N})$:

Notations are taken from Bloch-Vlasenko (last slide)

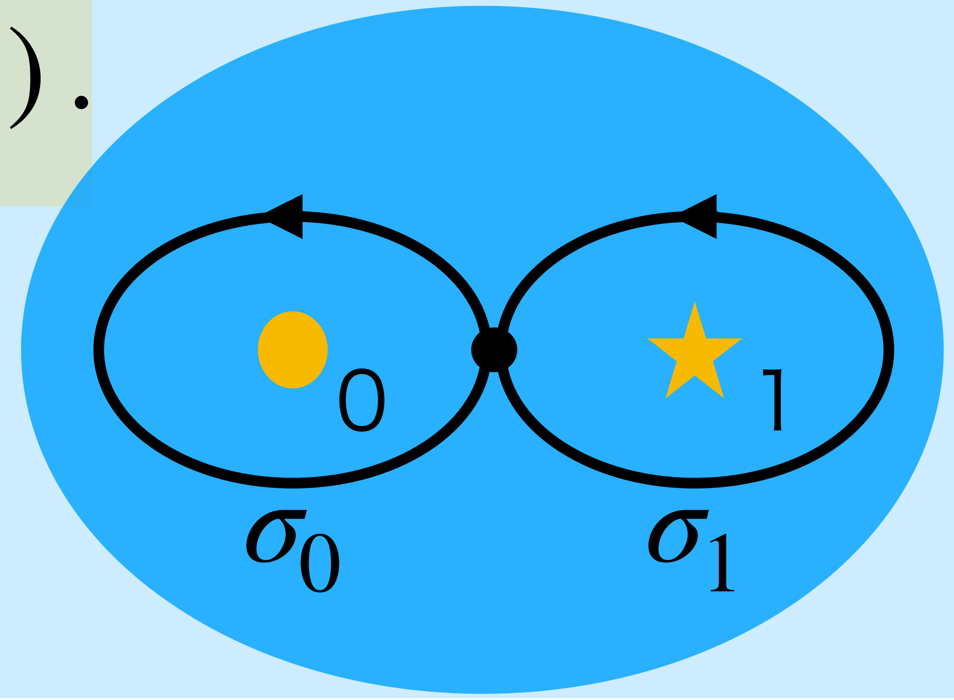
$$\Gamma_\xi(s) = \sum_j e^{2\pi i s n_j} \int_{\sigma_j} \langle v, \varepsilon_j \rangle x^s \frac{dx}{x} \quad \left(\xi \sim \sum_j \sigma_j \otimes \varepsilon_j \otimes e^{2\pi i s n_j} \in H_1(U^{\mathrm{an}}, \mathscr{E}^\vee \otimes x^s), v \otimes \frac{dx}{x} \in E \otimes \Omega^1_U \right)$$

Example $f\colon \mathbb{P}^1_y \to \mathbb{P}^1_x$, $x = f(y) = 1 - y^2$. $f^\circ : C^\circ = \mathbb{P}^1 \backslash \{0,1,-1,\infty\} \to U = \mathbb{P}^1_x \backslash \{0,1,\infty\}$.

$f^\circ_* \mathcal{O}_{C^\circ} = \mathcal{O}_U \oplus \mathcal{O}_U[y] \supset \mathcal{O}_U[y] =: E, \nabla[y] = -2^{-1}(1-x)^{-1}[y]dx, \mathcal{N} = (E, \nabla)$.

$\longrightarrow \mathfrak{M}(\mathcal{N})_\infty \simeq \mathscr{B}_{3/2}$.

$$\Gamma_\xi(s) = \int_{\sigma_1 \sigma_0 \sigma_1 \sigma_0^{-1}} (1-x)^{1/2} x^s \frac{dx}{x} = 2(1 - e^{2\pi i s}) \frac{\Gamma(s)\Gamma(3/2)}{\Gamma(s+3/2)}.$$

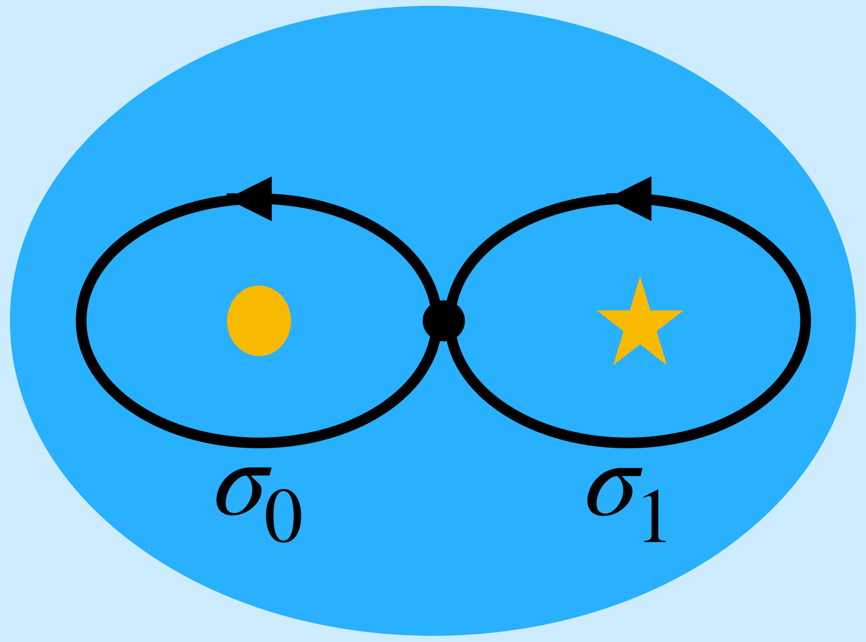


An approach to the question

Example $\mathscr{O}_U[y] =: E, \nabla[y] = -2^{-1}(1-x)^{-1}[y]dx, \mathscr{N} = (E, \nabla)$.

$\longrightarrow$ $\mathfrak{M}(\mathscr{N})_\infty \simeq \mathscr{B}_{3/2}$.

$$\Gamma_\xi(s) = \int_{\sigma_1\sigma_0\sigma_1\sigma_0^{-1}} (1-x)^{1/2}x^s\frac{dx}{x} = 2(1-e^{2\pi i s})\frac{\Gamma(s)\Gamma(3/2)}{\Gamma(s+3/2)}.$$



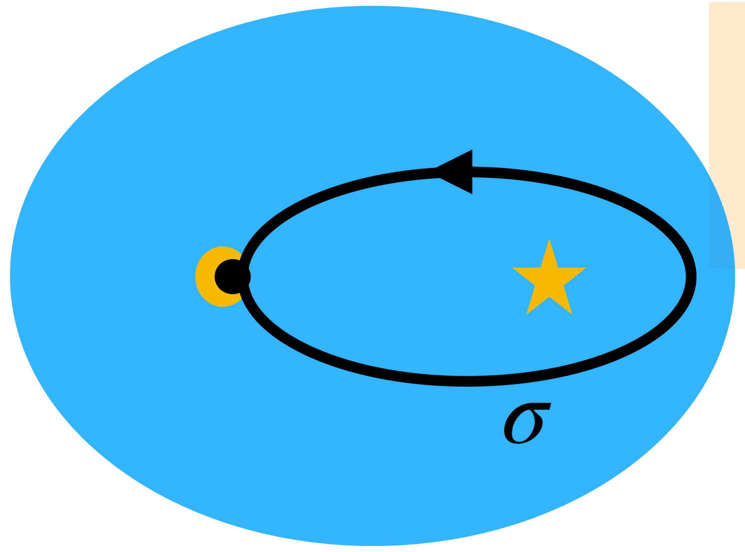
Regular singular modules Take $\alpha \in \mathbb{C}\backslash\mathbb{Z}$ and set $\mathscr{B}_\alpha := (K, (1+\alpha t)\phi)$.

$$\mathrm{Per}(\mathscr{B}_\alpha)_{|U} = \begin{cases} \mathscr{A}_{\mathrm{per}|U}\Gamma(s)/\Gamma(s+\alpha) & (e^{\pi i} \notin U), \\ \mathscr{A}_{\mathrm{per}|U}(1-u)\Gamma(s)/(1-e^{2\pi i\alpha}u)\Gamma(s+\alpha) & (e^0 \notin U). \end{cases}$$

To give a section with good asymptotic behavior, we take ‘rapid decay’ paths:

$\Gamma(s)/\Gamma(s+3/2)$

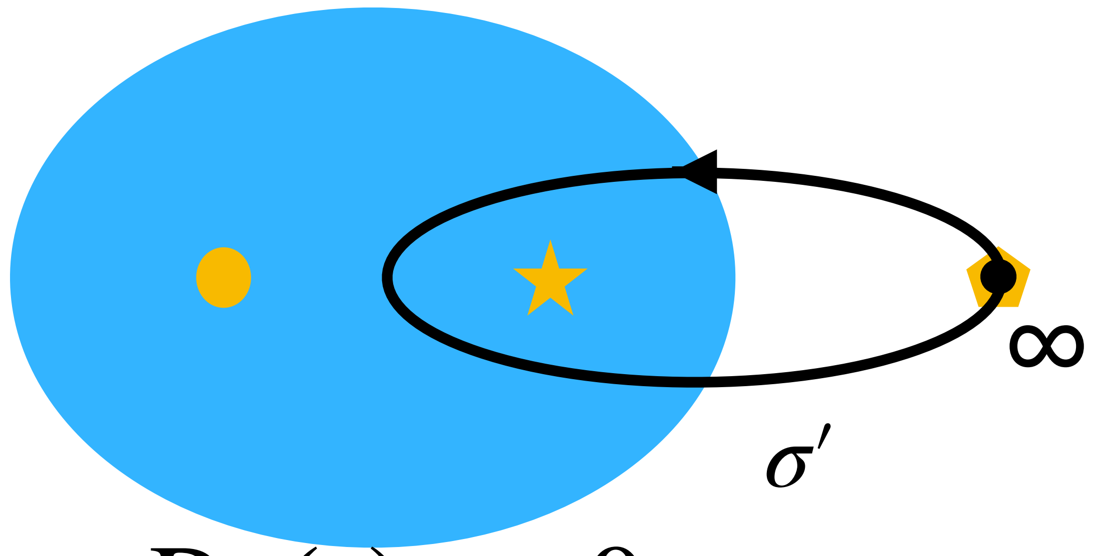
$$= (\mathrm{constant})\int_\sigma (1-x)^{1/2}x^s\frac{dx}{x}$$



$\mathrm{Re}(s) \gg 0$

$(1-u)\Gamma(s)/(1+u)\Gamma(s+3/2)$

$$= (\mathrm{constant})\int_{\sigma'} (1-x)^{1/2}x^s\frac{dx}{x}$$



$\mathrm{Re}(s) \ll 0$

Motivic Gamma functions

Recently, there appear interesting studies on Mellin transformations:

Golyshev, Vasily V., and Don Zagier. "Proof of the gamma conjecture for Fano 3-folds of Picard rank 1." *Izvestiya: Mathematics* 80.1 (2016): 24.

§2.4. Higher Frobenius limits: beyond the gamma conjecture.

Spencer Bloch, Masha Vlasenko. "Gamma functions, monodromy and Frobenius constants." Communications in Number Theory and Physics 15 (2021), no. 1, 91–147

etc.

$f: Y \to \mathbb{C}_t$

$f_* \mathcal{O}_Y \quad t^{-1}$

$$\frac{s^r}{(1-e^{-2\pi i s})^{r-d}}\Gamma_{\xi_0}(s) = \sum_{n=d}^{\infty} \kappa_n s^n$$

motivic Gamma function

Frobenius constants are periods.