On attractor points in the moduli space of CY 3-folds

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§1 Geometry of the moduli space of Calabi-Yau threefolds

$f: \mathcal{X} \to \mathcal{M}$ universal family of smooth proj. Calabi-Yau 3 folds of fixed diffeomorphism type.

$(R^3 f_* \mathbb{Z}, \mathcal{F}^\bullet, Q)$ polarized VHS with a canonical flat connection ∇.

$\mathcal{L} = \mathcal{F}^3 \to \mathcal{M}$ holom. line bdl. (Hodge bundle)

$\mathcal{M}$ carries the structure of a projective special Kähler manifold.

Sections of $\mathcal{L}$: nowhere vanishing holom. 3 form Ω

HR bilinear relation $\Rightarrow$

$\langle \Omega, \bar{\Omega} \rangle := i \int_X \Omega \wedge \bar{\Omega}$ hermitian metric on $\mathcal{L}$

view Ω as a local fct on $\mathcal{M}$.

$e^{-K} = \langle \Omega, \bar{\Omega} \rangle$ define a Kähler metric on $\mathcal{M}$ $\quad g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$. Weil-Petersson metric

Kodaira-Spencer theory $\Rightarrow T_s \mathcal{M} \cong H^{2,1}(X_s)$

$\Rightarrow$ basis for $H^{2,1}(X_s)$: $\chi_i = e^{K/2}\left(\partial_i \Omega - \frac{\langle \partial_i \Omega, \bar{\Omega} \rangle}{\langle \Omega, \bar{\Omega} \rangle} \Omega\right)$

$\Rightarrow g_{i\bar{j}} = -i \langle \chi_i, \bar{\chi}_j \rangle$

§2 Attractor points

Physics: $N=2$ supergravity on $\mathbb{R}^{1,3}$ with $h^{2,1}(X)$ vector multiplets, parametrizing $\tilde{\mathcal{M}}$

Scalar fields in the vector multiplets define a non linear σ model $z: \mathbb{R}^{1,3} \to \tilde{\mathcal{M}}$

Supersymmetric, static, spherically symmetric black hole solutions characterized by electric, magnetic charge $\gamma \in H_3(X, \mathbb{Z})$

mass: $m^2 = |Z(\gamma; z)|^2$

where $Z(\gamma; z) := e^{K/2} \int_\gamma \Omega$

D-brane central charge function (function on $\tilde{\mathcal{M}}$)

Ferrara, Kallosh, Strominger '95

There is a gradient flow on $\tilde{\mathcal{M}}$, called attractor flow.

$$\mu \frac{\partial}{\partial \mu} z^i = - g^{i\bar{j}} \partial_{\bar{j}} \log |Z(\gamma; z(\mu))|^2$$

The fixed points z_* of this flow are called attractor points.

§3 Hodge-theoretic formulation

<u>Thm</u> (Moore '98)

$|Z(\gamma;z)|^2$ has stationary point at $z = z_*(\gamma) \in \tilde{\mathcal{M}}$

with $Z(\gamma, z_*) \neq 0 \iff$ (*)

$PD(\gamma) \in H^{3,0}(X_{z_*}) \oplus H^{0,3}(X_{z_*}) \cap H^3(X_{z_*}, \mathbb{Z})$

If $z_*(\gamma)$ exists in the interior of $\tilde{\mathcal{M}}$, then it is a local minimum of $|Z(\gamma, z)|^2$.

Interpretation of (*)

$V_z = H^{3,0}(X_z) \oplus H^{0,3}(X_z)$ complex 2-plane in $H^3(X, \mathbb{C})$

$V_{z,\mathbb{R}} = V_z \cap H^3(X, \mathbb{R})$ real 2-plane in $H^3(X, \mathbb{R})$

$\Lambda_z = V_{z,\mathbb{R}} \cap H^3(X, \mathbb{Z})$

<u>Def</u>: $z \in \tilde{\mathcal{M}}$ is an attractor point of rank 1 or 2 if $\operatorname{rank} \Lambda_z = 1$ or 2.

Rank 2 attractor points are rare:

First non-trivial example:

Candelas - de la Ossa - Elmi - van Straten (2019)

Simplifying assumption, $h^{2,1}(X)=1$.

$$\Lambda_{\mathbb{Z}}\otimes\mathbb{C} = H^{3,0}(X_z)\oplus H^{0,3}(X_z)$$

$$\Lambda_{\mathbb{Z}}^{\perp}\otimes\mathbb{C} = H^{2,1}(X_z)\oplus H^{1,2}(X_z)$$

$\Lambda_{\mathbb{Z}}\oplus\Lambda_{\mathbb{Z}}^{\perp}\subset H^3(X,\mathbb{Z})$ of finite index

$$\Rightarrow H^3(X,\mathbb{Q}) = \underbrace{\Lambda_{\mathbb{Z},\mathbb{Q}}}_{H_1}\oplus\underbrace{\Lambda^{\perp}_{\mathbb{Z},\mathbb{Q}}}_{H_2}$$

H_1: HS of weight 3, type $(1,0,0,1)$

$\Rightarrow$ HS of a rigid CY 3-fold

H_2: HS of weight 3, type $(0,1,1,0)$

$H_2\otimes\mathbb{Q}(1)$: " " 1, type $(1,1)$

$\Rightarrow$ HS of an elliptic curve

§4 Interlude: Modularity of elliptic curves

1) Thm (Wiles – Taylor et al.)
$E/\mathbb{Q}$ elliptic curve is modular:
$\exists f \in S_2(\Gamma_0(N))$ Hecke eigenform
N conductor of E s.t. $L(E,s) = L(f,s)$

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda_f, \quad \Lambda_f = \frac{1}{2\pi i} \int_{H_1(X_0(N),\mathbb{Z})} f(z)dz$$

modular parametrization

$$\begin{aligned} X_0(N) &\longrightarrow E \\ [\tau] &\longmapsto P_\tau = \int_\infty^\tau f(z)dz \mod \Lambda_f \\ \infty &\longmapsto 0 \end{aligned}$$

Manin: $\Lambda_f = \mathbb{Z}\omega_f^+ \oplus \mathbb{Z}\omega_f^-$
$\omega_f^\pm$ are periods of f.

2) Let $\mathcal{E} \to X_0(N)$ be a universal family of ell. curves with cyclic subgroup of order N
$\exists$ special pts $[\tau] \in X_0(N)$, Heegner points, at which $\mathrm{End}_K \mathcal{E}_{[\tau]} \neq \mathbb{Z}$

1) $\tau \in \mathbb{Q}(\sqrt{-D})$, $D>0$, $j(\tau) \in \overline{\mathbb{Q}}$

2) $L/\mathbb{Q}$ number field, E/L elliptic curve

Assume E has CM by $\mathcal{O}_K$ of $K \subset L$

Then $L(E/L, s) = L(s, \psi_{E/L}) L(s, \overline{\psi}_{E,L})$

$\psi_{E/L}$ Hecke Grössencharacter.

Modularity of rigid CY 3 folds ($h^{2,1}=0$)

<u>Thm</u>: (Gouvea, Yui, Dieulefait)

Let $X/\mathbb{Q}$ be a rigid smooth proj. CY 3 fold. Then X is modular, i.e.

$\exists N, f \in S_4(\Gamma_0(N))$ s.t.

$L(X,s) = L(f,s)$.

§5 Arithmetic properties of attractor points

For simplicity $h^{2,1} = 1$

Conjecture (Deligne, Golyshev-Zagier, Bönisch, Klemm, S, Zagier)

let $\Pi_{\mathbb{Z}}$ be the period matrix of $f: \mathcal{X} \to \mathcal{M}$ in an integral symplectic basis.

Let Π_* be the period matrix of X_{z_*}

let $\Pi_* = M_* \Pi_{\mathbb{Z}}$

Then $\exists\ N_1, N_2,\ f \in S_4(\Gamma_0(N_1))$
$g \in S_2(\Gamma_0(N_2))$

and a choice of basis of $H^3(X, \mathbb{Z})$ s.t.

$$M_* = \begin{pmatrix} \begin{matrix} \omega_f^+ & \omega_f^- \\ \eta_F^+ & \eta_F^- \end{matrix} & 0 \\ 0 & \begin{matrix} \tilde{\omega}_g^+ & \tilde{\omega}_g^- \\ \tilde{\eta}_G^+ & \tilde{\eta}_G^- \end{matrix} \end{pmatrix}$$

where $\omega_f^{\pm}$ are periods of f, $\eta_F^{\pm}$ are the quasi periods of a meromorphic partner F of f. Similarly for g, G s.t.

$$L(X_*, s) = L(f, s)\, L(g, s)$$

§6 Examples

Consider hypergeometric families of CY 3-folds

Assume ∇ has 3 regular singularities

$\mathcal{M} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $\overline{\mathcal{M}} = \mathbb{P}^1$

Thm: (Doran, Morgan)

$\exists$ 14 $\mathbb{Q}$-VHS of Hodge type $(1,1,1,1)$

s.t. $(T_0 - 1)^4 = 0$, $(T_0 - 1)^3 \neq 0$

$(T_1 - 1)^2 = 0$, $(T_1 - 1) \neq 0$

T_z : local monodromy of ∇ around $z \in \overline{\mathcal{M}}$

$\nabla \pi = 0 \rightsquigarrow$ hypergeometric diff. eqn. $({}_4F_3)$

$$\left(\theta^4 - z \prod_{i=1}^{4} (\theta - \alpha_i)\right) \pi = 0, \quad \theta = z \frac{d}{dz}$$

z local parameter near $0 \in \overline{\mathcal{M}}$.

Famous example (Candelas, de la Ossa, Green, Parkes '91)

$\alpha_i = \frac{i}{5}$, $i = 1, 2, 3, 4$.

Conjecture (Bönisch, Klemm, S, Zagier)

$\alpha = \left(\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}\right)$, $z = 2^{-4} 3^{-3}$, $N_1 = 180$, $N_2 = 36$

$\alpha = \left(\frac{1}{3} \frac{1}{3} \frac{2}{3} \frac{2}{3}\right)$, $z = 2^{-3} 3^{-6}$, $N_1 = N_2 = 54$

Then the conjecture holds with explicitly given f, F, g, G.

Evidence: Numerically verified to very high precision (≈ 100s of digits)