The quantum GIT conjecture

Daniel Pomerleano

July 14, 2023

Based on:

- work **in progress** with Constantin Teleman

Throughout G is a connected compact Lie group, $G_{\mathbb{C}}$ is its complexification.

Kirwan surjectivity

Let M be a symplectic manifold with Hamiltonian G-action and moment map $\mu : M \longrightarrow \mathfrak{g}^*$. For simplicity, assume that G acts freely on $\mu^{-1}(0)$ so that we can form

$$M//G := \mu^{-1}(0)/G.$$

Theorem (Kirwan)

The natural map

$$\kappa : H^*_G(M) \longrightarrow H^*(M//G) \tag{1}$$

is surjective.

Kirwan surjectivity(cont)

The idea of the proof is to use the function $f = ||\mu||^2 : M \longrightarrow \mathbb{R}$ as a "Morse" function on M, where $||\mu||$ is the norm associated to any invariant inner product on $\mathfrak{g}$. M acquires a "Kirwan stratification" by descending manifolds

$$M := \bigcup_{\beta} S_\beta \tag{2}$$

Theorem (Kirwan)

The "Morse" function $||\mu||^2$ is equivariantly perfect i.e. the spectral sequence

$$\bigoplus_{\beta} H^*_G(S_\beta) => H^*_G(M) \tag{3}$$

collapses.

Kernel of the Kirwan map

It is interesting to describe the kernel of κ. Consider the simplest case when $G = S^1$. Define

$$K_\pm := \{\alpha \in H^*_{S^1}(M, \mathbb{Q}) | \alpha_{|F \cap M_\pm} = 0\} \tag{4}$$

where $M_+ := \mu^{-1}(0, \infty)$, $M_- := \mu^{-1}(-\infty, 0)$.

Theorem (Tolman-Weitsman)

The kernel of κ is given by: $K := K_- \oplus K_+$.

Remark

For G a torus, we have

$$ker(\kappa) = \sum_S (K_-^S \oplus K_+^S) \tag{5}$$

where S runs over "generic" circles.

3D TQFT Yoga

Slogan (Teleman '14)

A compact symplectic manifold M with Hamiltonian G-action should define an object $\mathbb{O}_G(M)$ in the Rozansky-Witten 2-category (or 3D B-model) of the "BFM space" $\mathrm{Spec}(\mathcal{C}_{G_{\mathbb{C}}})$.

- More recently, [Bullimore, Dimofte, Gaiotto] and [Teleman] proposed that one can more generally associate an object in the B-model of the Coulomb branch with matter, $\mathrm{Spec}(\mathcal{C}_{G_{\mathbb{C}}}(V))$, for any complex representation V of $G_{\mathbb{C}}$.

3D TQFT Yoga (cont)

- An object of the 2-category is roughly speaking expected to be a holomorphic Lagrangian $\mathcal{L} \subset \mathrm{Spec}(\mathcal{C}_{G_{\mathbb{C}}}(V))$ together with a sheaf of categories $\mathcal{C}$ over $\mathcal{L}$.

Thus, we expect

$$G \curvearrowright M => \mathcal{L}_{M,G}(V) \subset \mathrm{Spec}(\mathcal{C}_{G_{\mathbb{C}}}(V)).$$

Goals of Talk

- *Explain implications of this picture for equivariant quantum cohomology and quantum cohomology of GIT quotients.*

Definition

The algebra $\mathcal{C}_{G_{\mathbb{C}}}$ is defined to be the vector space $\hat{H}_*^G(\Omega G, \mathbb{C})$ equipped with the Pontryagin product.

Two basic geometric facts concerning Spec($\mathcal{C}_{G_{\mathbb{C}}}$)) are the following:

1. $\hat{H}_*^G(\Omega G, \mathbb{C})$ is a Hopf algebra over $H^*(BG, \mathbb{C})$. As a consequence, Spec($\mathcal{C}_{G_{\mathbb{C}}}$)) has the structure of a group scheme over Spec($H^*(BG, \mathbb{C})$).
2. The spectrum Spec($\mathcal{C}_{G_{\mathbb{C}}}$) is a smooth holomorphic symplectic manifold. This is due to the existence of the quantization $\hat{H}_*^{S^1 \times G}(\Omega G, \mathbb{C})$.

Example: G=SU(2)

Example

- Take $(\mathbb{C} \times \mathbb{C}^*)/\mathbb{Z}/2\mathbb{Z}$ where the $\mathbb{Z}/2\mathbb{Z}$-action identifies (h, z) with $(-h, z^{-1})$.
- The Coulomb branch is given by blowing this up at $(0, 1)$ and then removing the proper transform of the zero-section $\{0\} \times \mathbb{C}^*/\mathbb{Z}/2\mathbb{Z}$.

Quantum cohomology

Let (M^{2n}, ω) be a monotone closed symplectic manifold ($[\omega] = [c_1(M)] \in H^2(M)$), equipped with a Hamiltonian action of G.

- Let $QH^*_{S^1 \times G}(M)$ denote the quantum cohomology of M which is equivariant with respect to the G-action and loop rotation. As a vector space this is given by

$$QH^*_{S^1 \times G}(M) := H^*_G(M)[q^{\pm 1}, u]$$

where q is the Novikov variable, and u is the positive generator of $H^*(BS^1)$.

This vector space carries much structure, the most elementary pieces of which are as follows:

- The reduction modulo u is the ordinary quantum cohomology $QH^*_G(M)$, which carries an equivariant quantum product.
- The full equivariant quantum cohomology $QH^*_{S^1 \times G}(M)$ carries a quantum connection $\nabla_{q\partial_q}$, which differentiates in the direction of the Novikov variable.

Module structures

Theorem (Gonzalez-Mak-P '22)

There is a module action

$$\mathcal{S} : \hat{H}_*^{S^1 \times G}(\Omega G) \otimes QH^*_{S^1 \times G}(M) \longrightarrow QH^*_{S^1 \times G}(M) \qquad (6)$$

Corollary

The support of $QH^*_G(M)_{|q=1}$ as a coherent sheaf over $BFM(G^\vee_{\mathbb{C}})$ is a (possibly singular) holomorphic Lagrangian subvariety $\mathcal{L}_G(M) \hookrightarrow \mathcal{Z}_{G^\vee_{\mathbb{C}}}$.

Remark

This result uses Gabber's famous result on the " involutivity of characteristics" for modules over a deformation quantization.

Examples

- Let M be a monotone toric variety acted on by T. There is a combinatorially defined "Hori-Vafa" superpotential $W_{HV} : T^{\vee}_{\mathbb{C}} \longrightarrow \mathbb{C}$. Then $\mathcal{L}_G(M)$ is given by

$$\mathrm{graph}(dW_{HV}) \subset T^* T^{\vee}_{\mathbb{C}}.$$

- Let $M = G/T$, then there is an embedding of the classical Toda system $T^* T^{\vee}_{\mathbb{C}} \hookrightarrow \mathcal{Z}_{G^{\vee}_{\mathbb{C}}}$ (this involves an alternative "Toda" realization of $\mathcal{Z}_{G^{\vee}_{\mathbb{C}}}$ as a Hamiltonian reduction of $T^* G^{\vee}_{\mathbb{C}}$ by $N^{\vee} \times N^{\vee}$). Then $\mathcal{L}_G(G/T)$ is given by a cotangent fiber in $T^* T^{\vee}_{\mathbb{C}}$.

Seidel operators

The starting point is a classical construction of Seidel. For simplicity, let σ be a co-character $\sigma : S^1 \longrightarrow T$ (at this point could be an arbitrary element of $\Omega Ham(M, \omega)$ but not later on). We obtain a fiber bundle $E(\sigma) \longrightarrow \mathbb{C}P^1$ by gluing two copies of a disc

$$D_0^2 \times M \bigsqcup D_\infty^2 \times M / \sim$$

$$(x, e^{2\pi i\theta}) \sim (\sigma(\theta)x, e^{2\pi i\theta})$$

The divisors at $0, \infty$ are canonically diffeomorphic to M. Given a section class $A_\sigma \in H_2(E(\sigma), \mathbb{Z})$ we can form the moduli of two pointed sections $\bar{\mathcal{M}}_{0,2}(E(\sigma), A_\sigma)$. Using these moduli spaces, we can define a "push-pull operation"

$$Z \longrightarrow ev_{\infty,*}[Z \times_{ev_0} \bar{\mathcal{M}}_{0,2}(E(\sigma), A_\sigma)]q^{c_1^{vert}(A_\sigma)}$$

which gives rise to a $\mathbf{k}[q^{\pm}]$-linear operator

$$S_\sigma^{(0)} : QH^*(M) \longrightarrow QH^*(M)$$

Shift operators

The algebraic properties of Seidel operators are

1. $S_{\sigma_1} \cdot S_{\sigma_2} = S_{\sigma_1+\sigma_2}$.
2. The map $\sigma \longrightarrow S_\sigma(1)$ induces a ring homomorphism $\mathbf{k}[\mathcal{X}(T)] \longrightarrow QH^*(M)$.

Later on [Okounkov-Maulik] used the same idea to define shift-operators

$$S_\sigma : QH^*_{S^1\times T}(M) \longrightarrow QH^*_{S^1\times T}(M).$$

These have slightly different algebraic properties:

1. S_σ is a "σ-twisted" homomorphism (with respect to the equivariant parameters).
2. S_σ commutes with the quantum connection.

Quantum GIT conjecture

Definition

A moment map will be called balanced if

1. $[c_1^G(TM)] = [\omega^G]$, where ω^G is the closed equivariant extension of ω determined by the moment map.
2. G-acts freely on $\mu^{-1}(0)$.

In this case, $M//G$ is again monotone. We can ask:

Question

Supposed μ is balanced. Is there a formula for $QH^*(M//G)$ in terms of $QH^*_G(M)$ and the action by non-abelian Seidel operators?

Let $L_{\mathbf{id}}$ denote the unit section of the group scheme structure on $\mathrm{Spec}(\mathcal{C}_{G_{\mathbb{C}}})$.

Conjecture (Teleman '14)

The quantum cohomology of a balanced symplectic quotient

$$QH^*(M//G) \cong QH^*_G(M) \otimes_{\hat{H}^G_*(\Omega G)} \Gamma(\mathcal{O}_{L_{\mathbf{id}}}) \tag{7}$$

When $G = (S^1)^r$, this concretely says that

$$QH^*(M//G) \cong \frac{QH^*_G(M)}{(z_i = 1)}$$

where $z_1, \cdots, z_r$ are the Seidel operators.

Batyrev's formula

Consider a compact toric Fano variety realized as a balanced symplectic quotient $\mathbb{C}^n//T$. We view $QH_T^*(\mathbb{C}^n)$ as having generators $h_1 \cdots, h_n$ (modulo certain linear relations). Then for any character $\chi \in \mathcal{X}(T)$, we consider

$$QSR(\chi) := \prod_{j, h_j(\chi) \geq 0} h_j^{h_j(\chi)} - q^{s(\chi)} \prod_{j, h_j(\chi) \leq 0} h_j^{-h_j(\chi)} \tag{8}$$

Theorem (Batyrev, Givental)

$$QH^*(\mathbb{C}^n//T) \cong \frac{QH_T^*(\mathbb{C}^n)}{< QSR(\chi) >}. \tag{9}$$

Batyrev's formula (cont)

Instead consider

$$SH_T^*(\mathbb{C}^n) := QH_T^*(\mathbb{C}^n)[s_\Delta^{-1}] \tag{10}$$

Then for any character $\chi \in \mathcal{X}(T)$, we have a Seidel element

$$S(\chi) := \prod_j (q^{-1}h_j)^{h_j(\chi)} \tag{11}$$

$$QH^*(\mathbb{C}^n//T) \cong \frac{SH_T^*(\mathbb{C}^n)}{< S(\chi) = 1 >}. \tag{12}$$

Main result

Theorem (P-Teleman, in progress)

- *The formula* (7) *holds.*
- *Given an* T *action on* M *with balanced moment map,* $\mu : M \longrightarrow \mathbb{R}$ *then* $QH^*_T(M)$ *is a free module over* $\mathbb{C}[q^{\pm}, z_i^{\pm}]$ *with rank* $\dim(H^*(M//T, \mathbb{C}))$.

To keep things more explicit, we consider the abelian case, $G = T$. The proof has two steps:

- An additive argument in Hamiltonian Floer cohomology, borrowing ideas from Borman-Sheridan-Varolgunes.
- Using Lagrangian correspondences to construct a ring homomorphism (where the Seidel elements manifestly act trivially).

Idea of additive argument

The basic idea of the proof is to use T-equivariant Hamiltonian Floer cohomology of $H_K := \frac{1}{2}K||\mu||^2$ as $K \longrightarrow \infty$.

- The Floer complexes are all isomorphic in that there are natural isomorphisms $CF_T^*(M; H_K) \cong CF_T^*(M; H_{K'})$ (and indeed these are all isomorphic to T-equivariant $QH_T^*(M)$.)
- However the time-one periodic orbits of the Hamiltonian vector field change quite a bit, indeed there are more and more periodic orbits which appear near $\mu = 0$.
- So we as a first approximation take some $K_i \longrightarrow \infty$ and consider

$$CF_T^*(M; H) := \text{hocolim}_i \, CF_T^*(M; H_i) \tag{13}$$

where $H_i = 1/2K_i||\mu||^2$.

- The problem is that this contains generators corresponding to other orbit sets other than the desired ones near zero (e.g. fixed points).
- So we want to filter this Floer complex so as to exclude these undesired orbits. The key to doing this is the so called monotone index of a capped periodic orbit which is defined to be

$$mix(x, [u]) = \deg(x, [u]) - 2A_{H_K}(x, [u]). \tag{14}$$

It is independent of the capping class $[u]$.

Lemma

For any $(x, [u]) \in \mathcal{X}(M; H_K)$, $mix(x, [u]) \geq K\mu^2 + C_0$ for some constant C_0 independent of K.

Take $\delta_i \longrightarrow 0$ such that $K_i\delta_i \longrightarrow \infty$. Let $\mathcal{F}_{\geq p}CF^*_{S^1}(M;H)$ be the subcomplex generated by orbits $(x,[u])$ which satisfy :

$$A_{H_i}(x,[u]) \geq p - K_i\delta_i.$$

Define

$$CF^*_T(M;H)^{(p)} := \sigma_{<p}\mathcal{F}_{\geq p}CF^*_T(M;H) \tag{15}$$

be the chains of degree $< p$. Set

$$\widetilde{CF}^*_T(M;H) := \mathrm{holim}_p\, CF^*_T(M;H)^{(p)} \tag{16}$$

Proposition

The cohomology of this complex is unchanged i.e. we still have:

$$H^*(\widetilde{CF}^*_T(M;H)) \cong QH^*_T(M). \tag{17}$$

- We put a q-adic filtration on $\widetilde{CF}^*_T(M;H)$ by choosing cappings so that the "Morse-Bott" CZ index has degree 0.
- A geometric argument shows that the Floer differential does not decrease this filtration.

Theorem

The q-adic filtration on $\widetilde{CF}^*_T(M;H)$ gives rise to a convergent spectral sequence with E_1 page

$$E_1 = H^*(M//T, \mathbb{Q}) \otimes \mathbb{C}[q^{\pm}, z_i^{\pm}]$$

and which converges to $QH^*_T(M)$. This spectral sequence collapses at E_1.

Ring homomorphism

Consider the moment Lagrangian correspondence $L_\mu \subset M \times M//G$ given by pairs of points:

$$(m, \bar{m}) \subset M \times M//G$$

Theorem (Fukaya)

There is an isomorphism:

$$F : HF^*_G(L_\mu, L_\mu) \cong QH^*(M//G) \tag{18}$$

Composing this with the closed open map gives :

$$F \circ CO : QH^*_G(M) \longrightarrow QH^*(M//G) \tag{19}$$

Putting things together

The remainder of the argument consists of three observations:

1. We show that $F \circ CO : QH^*_T(M) \longrightarrow QH^*(M//T)$ is surjective based on reduction to the classical Kirwan map.
2. The Seidel operators satisfy $z_i = 1$ on $HF^*_T(L_\mu, L_\mu)$. This is based off of the interpretation of Seidel operators in terms of Lagrangian mondromy.
3. By comparing ranks, this describes the entire kernel!

Fourier transform

For loop equivariant quantum cohomology, one expects a Fourier transform relationship:

$$FT : QH^*_{S^1 \times T}(M) \cong QH^*_{S^1}(M//G)[b_1^{\pm}, \cdots, b_r^{\pm}]. \qquad (20)$$

where $b_1, \cdots, b_r$ are Novikov variables in the direction of the Kirwan restriction of the equivariant parameters.

Conjecture

Trivialize $G \cong (S^1)^r$ and let $z_1, \cdots, z_r$ denote each of the shift operators. Then Woodward's quantum Kirwan map gives a map of the form (20) with

$$QK(z_i) = b_i.$$

Beyond Fano?

In the literature, one finds versions of Batyrev's formula for general compact toric varieties by suitably completing the equivariant quantum cohomology in the Novikov variables (FOOO, Iritani, Gonzalez-Woodward).

Question

What can be said when μ is not balanced? What about when M is not Fano?