Deformation theory, Fukaya's conjecture and the Gross-Siebert Program

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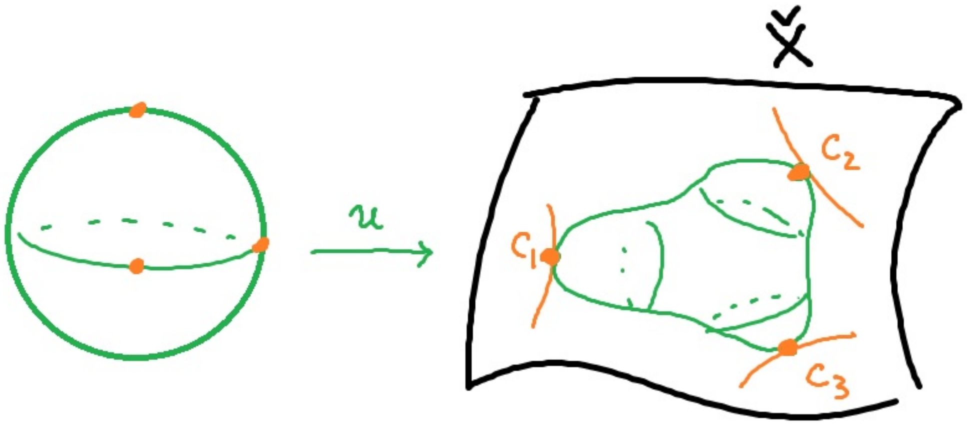
Mirror Symmetry

Calabi-Yau: $(\check{X}, \check{\omega}, \check{\Omega})$	Calabi-Yau: (X, Ω, ω)
locally $\check{X} \cong T^*\mathbb{R}^n,\ \check{\omega} = \sum_i d\check{x}_i \wedge d\check{y}_i$	**locally** $X \cong \mathbb{C}^n,\ \Omega = e^f dz_1 \wedge \cdots \wedge dz_n$
locally $X \cong \mathbb{C}^n,\ \Omega = e^f dz_1 \wedge \cdots \wedge dz_n$	**locally** $X \cong T^*\mathbb{R}^n,\ \omega = \sum_i dx_i \wedge dy_i$

Symplectic geometry



weighted count by $e^{-\int_S u^*(\check{\omega})} = \check{q}$

Complex geometry

Period integral : $\int_{C_i} \Omega_q = p_i(q)$

Mirror map

Mirror Symmetry

$(\check{X}, \check{\omega}, \check{\Omega})$ (X, Ω, ω)

Big Question:

1. Why!?
2. Given $\check{X}$, how to find X?

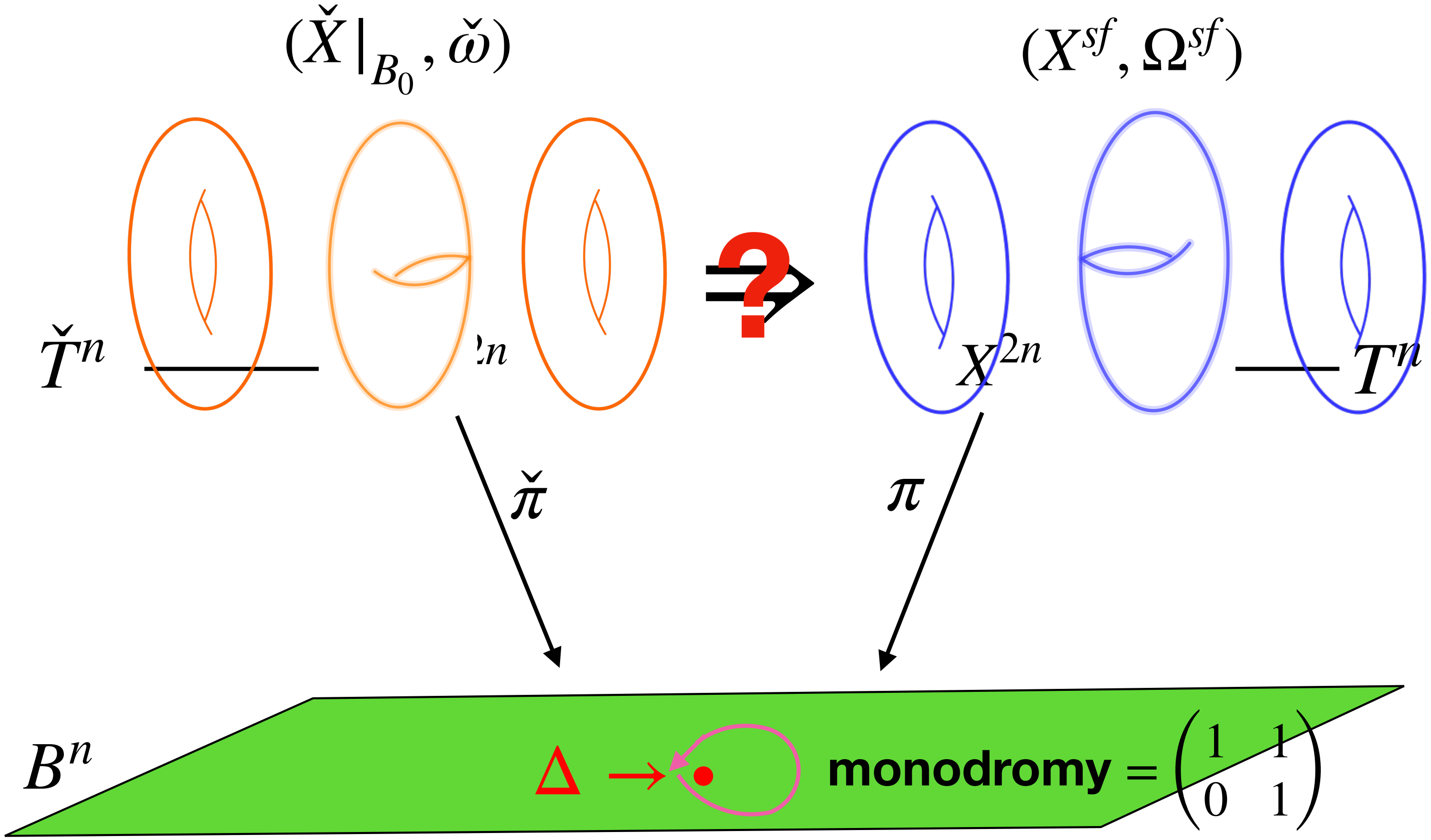
Idea :

Strominger-Yau-Zaslow conjecture

SYZ conjecture

Dual **1. Special** **2. Lagrangian** **torus fibration**

$\mathrm{Im}(\Omega)|_{T^n} = 0$ $\omega|_{T^n} = 0$

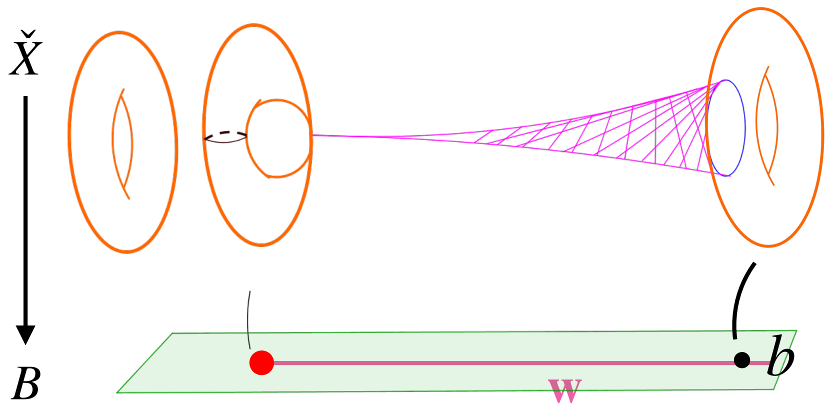


- $B_0 = B \backslash \Delta$: $\mathbb{Z}$**-affine manifold using Arnold-Liouville thm**

  i.e. translation in SL$(n, \mathbb{Z}) \ltimes \mathbb{R}^n$

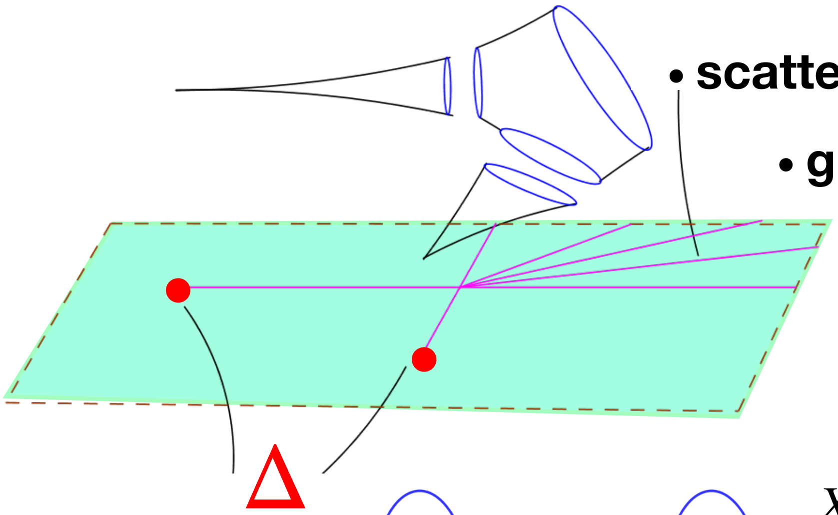
Fukaya's conjecture

1.



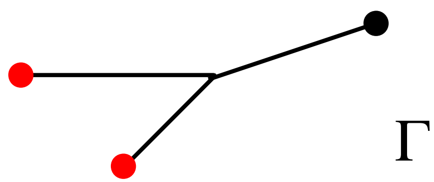
- holomorphic disk u, $\partial u \subset \check{T}_b$
- codimension-1 walls on B with disk on $\check{T}_b$

2.

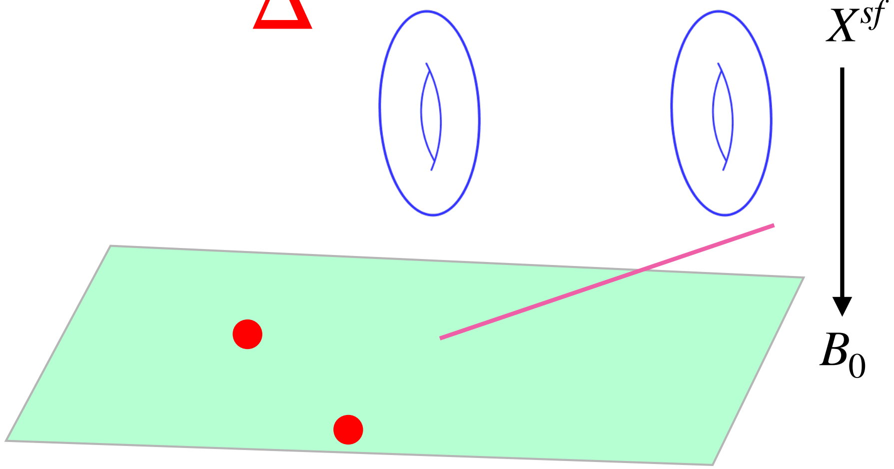


- scattered walls obtain by gluing with pair of pants
  - gradient trees Γ from Δ $\leftrightarrow$ holomorphic disks

e.g.



3.



- $\Gamma \rightsquigarrow \varphi_\Gamma \in \Omega^{0,1}(X^{sf}, T^{1,0})$
- supported on outgoing ray of Γ
- $\varphi = \sum_\Gamma \varphi_\Gamma$ satisfying Maurer-Cartan eqt.

$$\bar{\partial}\varphi + \frac{1}{2}[\varphi, \varphi] = 0$$

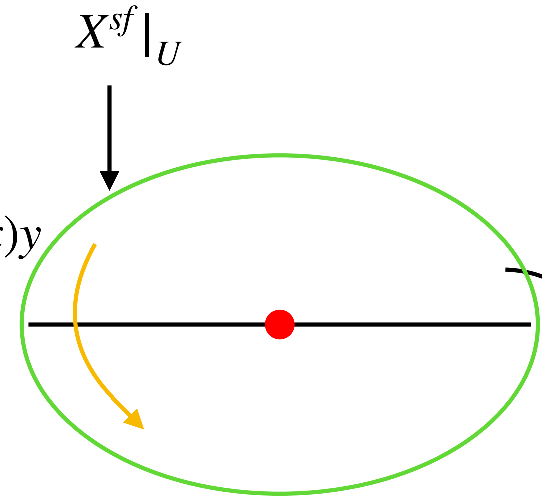
- $X^{sf} \rightsquigarrow X^{sf}_\varphi \subset X$

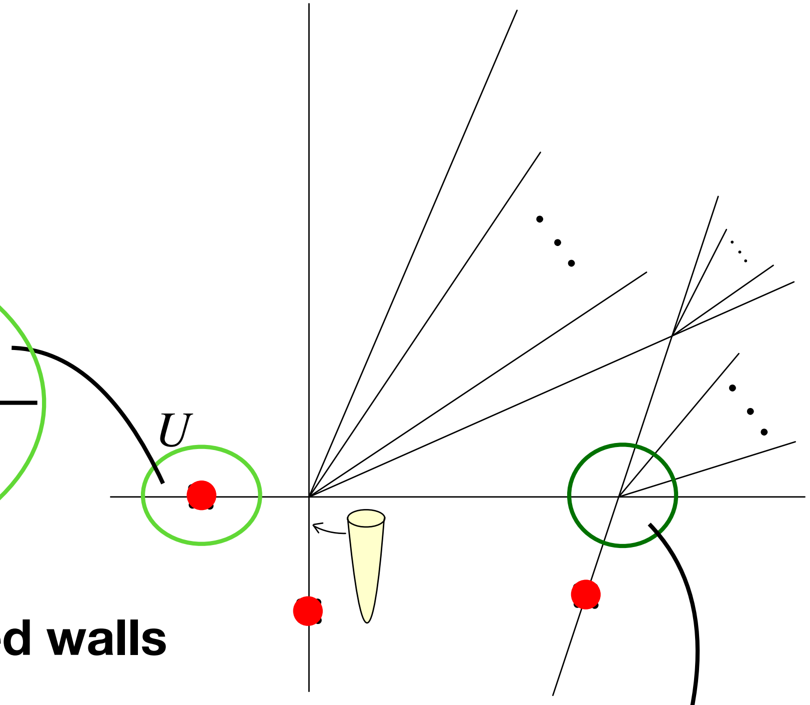
Scattering diagram

Kontsevich-Soibelman's combinatorial approach:

$\Theta_{\mathbf{w}}: \begin{array}{l} x \mapsto x \\ y \mapsto (1+x)y \end{array}$

$\in$ **Aut**$((\mathbb{C}^*)^2)$



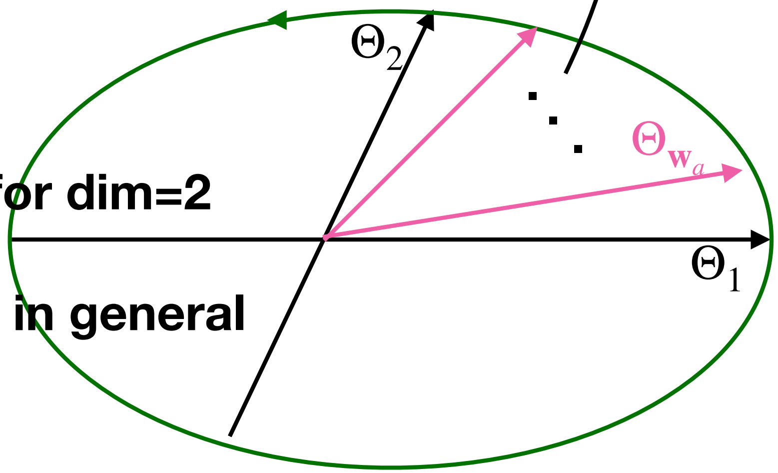


- **(K.-S.):** $\exists !$**ways to add new scattered walls Such that**

$$\Theta_2^{-1}\Theta_1^{-1}\Theta_2\Theta_1 \neq \text{id}$$

$$\prod_{\leftarrow} \Theta_{\mathbf{w}_a}$$

- **Thm (K.-S.):** (X, Ω) **can be constructed for dim=2**
- **Thm(Gross-Siebert):** (X, Ω) **constructed in general**



Fukaya's conjecture revisited

We consider the Maurer-Cartan eqt:

$$\boxed{\bar{\partial}\varphi + \frac{1}{2}[\varphi, \varphi] = 0} \qquad \textbf{in } \Omega^{0,*}(X^{sf}, T^{1,0})$$

Two questions :

Q1. How is the solution φ relate to scattering diagram?

Q2. What is a suitable "compactification" of $\Omega^{0,*}(X^{sf}, T^{1,0})$ as a dgLa?

$(\bar{\partial}, [\,\cdot\,,\,\cdot\,])$

Benefits :

Unified approach such that techniques of smooth manifolds can be applied!

Maurer-Cartan <—> Scattering

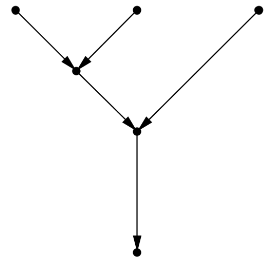
Partial Answer to Q1:

- **we work locally on** $U \subset B \backslash \Delta$

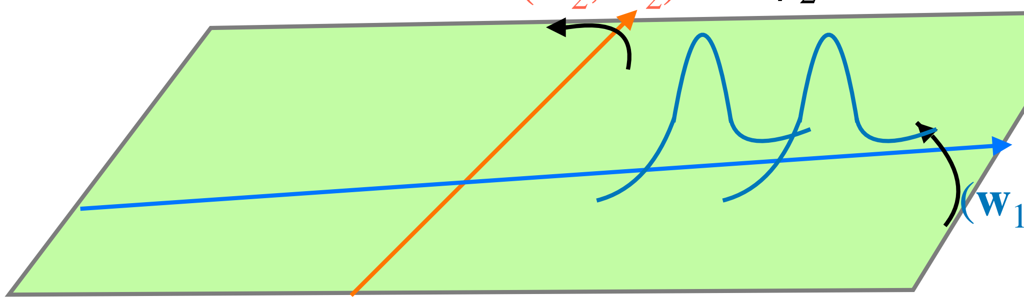
$X^{sf}|_U$

- $\varphi_{in} = \varphi_1 + \varphi_2$ **is Not a solution to MC equation**
- $\varphi = \varphi_{in} + \sum_T \varphi_T,\ T$ **: trivalent tree with one output**

e.g.



$(\mathbf{w}_2, \Theta_2) \rightsquigarrow \varphi_2$



$(\mathbf{w}_1, \Theta_1) \rightsquigarrow \varphi_1 = \delta_{\mathbf{w}_1, \lambda} \otimes \log(\Theta_1)$

$\in \Omega^{0,*}(X^{sf}, T^{1,0})$

Thm: (Chan-Leung- -)

As $\lambda \to \infty$, **we have**

$$\varphi \sim \varphi_{in} + \sum_{(\mathbf{w}_a, \Theta_a)} \delta_{\mathbf{w}_a, \lambda} \otimes \log(\Theta_a)$$

$(\mathbf{w}_a, \Theta_a)'s$ **are scattered walls in the K.-S. diagram** .

Q2 : What is the corrected dgLa?

Large complex structure limit (LCSL):

- **degenerating** $X_q^{sf} \rightsquigarrow X_0^{sf}$, X_0^{sf} **can be compactified as** X_0
- $X_0 \rightsquigarrow X$ **as smoothing/log deformation**

G.-S. : Use scattering diagram

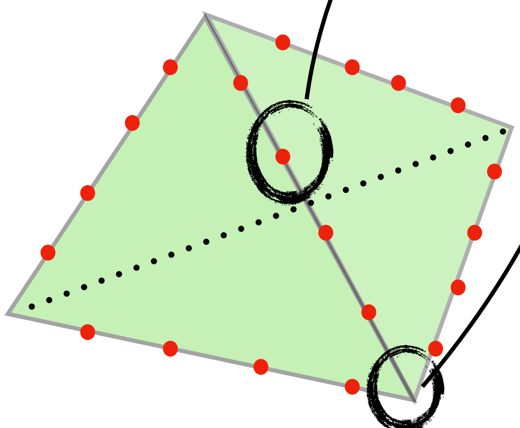
e.g. $X_q = \textbf{Zero}(\overbrace{q f(x_0, x_1, x_2, x_3) + x_0 x_1 x_2 x_3}) \subset \mathbb{P}^3$

$$X_0 = \bigcup_{i=0}^{3} \mathbb{P}_i^2$$

- **singularity of family** $= \{f(x) = 0\} \cap \bigcup_{i,j} \{x_i = x_j = 0\}$
- **locally:** $uv = q^l(1 + w) \subset \mathbb{C}^3 \times \mathbb{C}_w^*$

π_0

- **locally:** $\{xyz = q^l\} \subset \mathbb{C}^4$



- **G.-S.: construction of** X **from** X_0

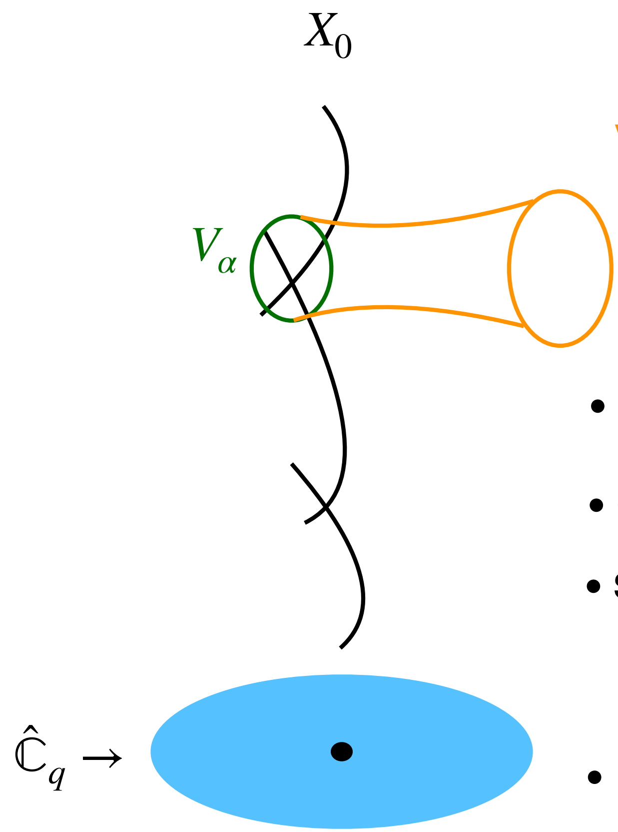
$\Updownarrow$

- **Consistent gluing of local models**

Using scattering diagram

dgBV approach to smoothing

- **analogue of "$\Omega^{0,*}(\wedge^* T^{1,0})$" for X_0 which is singular, we use algebraic construction**
- **idea of construction:**



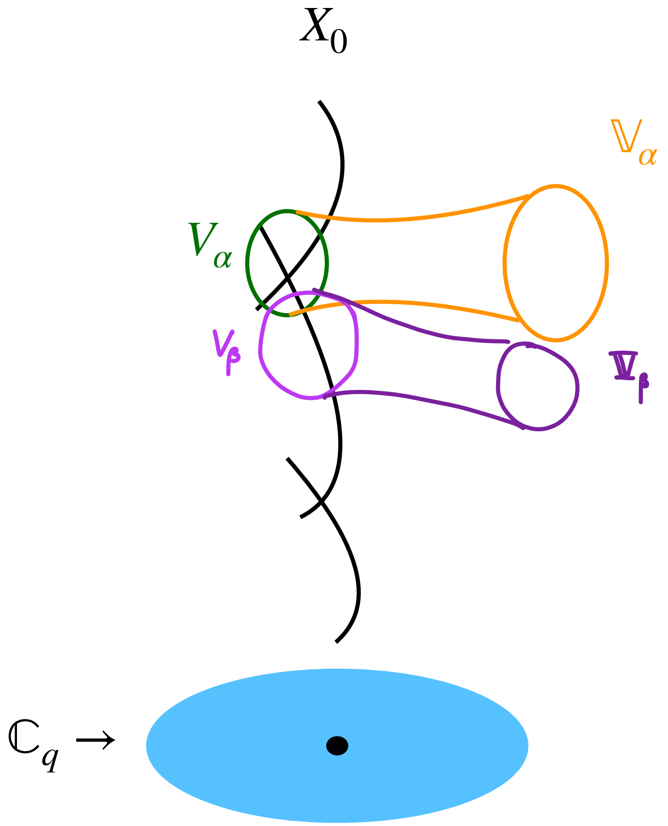
$\mathbb{V}_\alpha$ **-local models, e.g.:**
- $\{xyz = q^l\} \subset \mathbb{C}^4$
- $\{uv = q^l(1+w)\} \subset \mathbb{C}^3 \times \mathbb{C}^*_w$

- **scattering approach : solve for gluing** $g_{\alpha\beta} : \mathbb{V}_\alpha \to \mathbb{V}_\beta$
- **our approach : first dg resolve** $\mathbb{V}_\alpha$ **by** $PV^*(\mathbb{V}_\alpha) \sim$ **"**$\Omega^*(\mathbb{V}_\alpha, \wedge^* T^{1,0})$**"**
- **solve for gluing** $g_{\alpha\beta} : PV(\mathbb{V}_\alpha) \to PV(\mathbb{V}_\beta)$ ⟵ **Non-holo.! Easier**
- **get pre-dgBV** $PV(X_0)$**, remains to solve MC eqt.**

Thm: (Chan-Leung- -) **A pre-dgBV** $PV(X_0)$ **over** $\mathbb{C}[[q]]$ **can be constructed, and**

- **The MC eqt. can be solved with Hodge-de Rham degeneracy**
- **solution** φ **gives a (universal) smoothing** X_q **of** X_0 **over** $\mathbb{C}[[q]]$

Construction of pre-dgLa / C[[q]]



Step 1:

Take $\mathscr{G}^* = \wedge^* T_{\mathbb{V}_\alpha/\hat{\mathbb{C}}_q}$(**log**) **be the sheaf of BV algebra and resolve it as local sheaf of dgBV algebra** $(PV^*_\alpha, d_\alpha, \Delta_\alpha, \wedge)$

with partition of unity

Local Uniqueness of local model:

$r_{\alpha\beta,U}: \mathbb{V}_\alpha|_U \to \mathbb{V}_\beta|_U$, **for Stein** $U \subset V_\alpha \cap V_\beta$

Step 2:

Look for gluing as Genstenhaber algebra

$$g_{\alpha\beta} : PV^*_\alpha \to PV^*_\beta, \quad g_{\gamma\alpha} g_{\beta\gamma} g_{\alpha\beta} = id$$

Step 3:

Glue the operators

$$\begin{cases} d_\alpha + [\sigma_\alpha, \cdot\,] = g_{\beta\alpha} \circ (d_\beta + [\sigma_\beta, \cdot\,]) \circ g_{\alpha\beta} \\ \Delta_\alpha + [f_\alpha, \cdot\,] = g_{\beta\alpha} \circ (\Delta_\beta + [f_\beta, \cdot\,]) \circ g_{\alpha\beta} \end{cases}$$

We only require the identity $d^2 = \Delta^2 = d\Delta + \Delta d = 0 \quad (mod\ q)$

Perturbative approach towards semi-infinite VHS

From LCSL X_0

Ingredients:

$1.(\mathscr{E}_+, \nabla)$ **over the moduli** $\hat{\mathbb{C}}_q \times \hat{\mathbb{C}}_t$.

$\mathscr{E}_+ = H^*(PV^*, d + t\Delta + [\varphi(q,t), \cdot\,]), \varphi(q,t)$ **is a MC solution**

∇**: Gauss-Manin Connection with pole along q=0**

$2.$**An opposite subspace** $\mathscr{E}_-$ **to** $\mathscr{E}_+$

Notice that $\mathscr{E}_{0,1} = H^*(PV_0^*, d + \Delta)$ Res(∇)

$\mathscr{E}_-$ **defines by the weighted filtration defines by** $res(\nabla)$

$3.$**A pairing** $p : \mathscr{E}_{q,t} \times \mathscr{E}_{q,-t} \to \mathbb{C}$

choose a trace map $tr : \mathscr{E}_{0,1} \to \mathbb{C}$

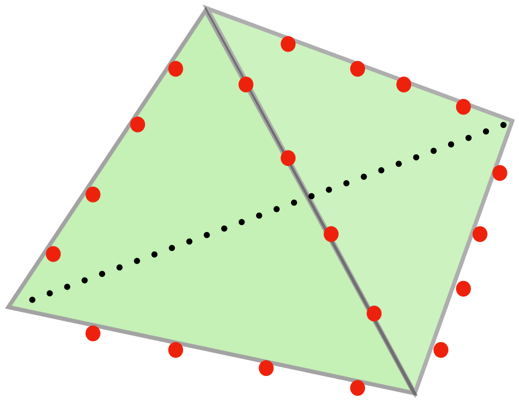
Thm: (Chan-Leung- -)

A miniversal sections ξ **of** $\mathscr{E}_+$ **can be solved perturbatively such that** $\xi e^{\varphi/t} - 1 \in \mathscr{E}_-$

Relation with Fukaya's conjecture

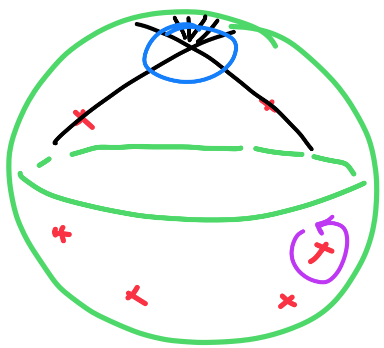
$X_0 = \bigcup_\sigma Y_\sigma$ **union of torics**

$\pi_0 = \bigcup_\sigma \mu_\sigma$ **generalized moment map**



$\check{B} = \bigcup_\sigma \sigma,$ **dual to B**

$\cong$



monodromy $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Integral affine with sing

Step 1:

$\pi_0^{-1}(W_\alpha) = V_\alpha$ **is a Stein cover**

Take $\mathscr{G}_\alpha^* = \pi_{0,*}(\wedge^* T_{\mathbb{V}_\alpha/\hat{\mathbb{C}}_q}(log))$

resolve it by $PV_\alpha^* = \Omega_{W_\alpha} \otimes \mathscr{G}_\alpha^*$

Step 2:

relate X_q^{sf} **with** X_q, **or their** PV^*

$\Psi : (PV^*|_{W_0}, d) \cong (PV_{sf}^*, d + [\varphi_{in}, \cdot\,])$

Step 3:

Thm: (Chan-Leung- -)

A scattering diagram $D(\varphi)$ **is extracted**

$D(\varphi)$ **is consistent**

Thank you!