A Categorical Description of Discriminants

Mirror Symmetry and Differential Equations, Boğaziçi University, Istanbul, Turkey
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based on joint work with Ludmil Katzarkov

Outline

Classical discriminants

The classical discriminant of a polynomial of degree $\leq n$

$$f(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n \in \mathbb{C}[x_1, \ldots x_n]$$

is a polynomial $\Delta(f) \in \mathbb{Z}[a_0, \ldots, a_n]$ such that $\Delta(f) = 0$ if f has a double root.

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$$\Delta(a_0 + a_1 x + a_2 x^2) = 4 a_0 a_2 - a_1^2$$

$$\Delta(a_0 + a_1 x + a_2 x^2 + a_3 x^3) =$$

$$27 a_0^2 a_3^2 + 4 a_0 a_2^3 + 4 a_1^3 a_3 - a_1^2 a_2^2 - 18 a_0 a_1 a_2 a_3$$

$$\begin{aligned}
&\Delta(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)\\
=&256 a_0^3 a_4^3 - 192 a_0^2 a_1 a_3 a_4^2 - 128 a_0^2 a_2^2 a_4^2 + 144 a_0^2 a_2 a_3^2 a_4 - 27 a_0^2 a_4^4\\
&+ 144 a_0 a_1^2 a_2 a_4^2 - 6 a_0 a_1^2 a_3^2 a_4 - 80 a_0 a_1 a_2^2 a_3 a_4 + 18 a_0 a_1 a_2 a_3^3\\
&+ 16 a_0 a_2^4 a_4 - 4 a_0 a_2^3 a_3^2 - 27 a_1^4 a_4^2\\
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\end{aligned}$$

Discriminants à la GKZ

$\mathcal{A}$-sets

- $\mathcal{A} = \{v_1, \ldots, v_n\}$ is a collection of elements of the lattice $N \cong \mathbb{Z}^d$, such that $\mathcal{A}$ generates N as a lattice
- There exists a group homomorphism $h : N \to \mathbb{Z}$ such that $h(v_i) = 1$ for any element $v_i \in \mathcal{A}$.

Notation:

$$Q := \mathrm{conv}(A) \subset \mathbb{R}^{d-1}, K := \sum_{1 \leq i \leq n} \mathbb{R}_{\geq 0} v_i = \mathbb{R}_{\geq 0} Q \subset \mathbb{R}^d$$

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- ∇_A is the Zariski closure in $\mathbb{C}^n$ of the set of polynomials $f = \sum_{1 \leq i \leq n} a_i x^{v_i}$ in $\mathbb{C}[x_1, \dots, x_d]$ such that there exists some $y \in (\mathbb{C}^\times)^n$ with the property that $f = 0$ is singular at y.
- The discriminant $\Delta_A \in \mathbb{Z}[a_1, \dots, a_n]$ is the irreducible polynomial (defined up to a sign) whose zero set is given by the union of the irreducible codimension 1 components of ∇_A. For the case codim $\nabla_A > 1$, one sets $\nabla_A = 1$.

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The principal A-determinant E_A is the polynomial in $\mathbb{Z}[a_1, \ldots, a_n]$ defined as

$$E_A := \prod_{\Gamma} (\Delta_{A \cap \Gamma})^{u(\Gamma) \cdot i(\Gamma)},$$

where the product is taken over all the non-empty faces Γ of the polytope $Q = \mathrm{conv}(A)$.

Let Γ be a non-empty face Γ of the polytope $Q = \mathrm{conv}(A)$.

- $i(\Gamma) := [N \cap \mathbb{R}\Gamma : \mathbb{Z}(A \cap \Gamma)]$ ($= 0$, if $A \cap \Gamma$ contains a basis of the restriction of N to the face determined by Γ)

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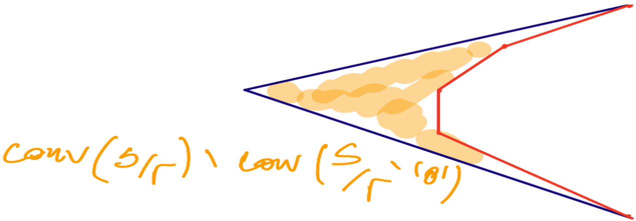
- $i(\Gamma) := [N \cap \mathbb{R}\Gamma : \mathbb{Z}(A \cap \Gamma)]$ ($= 0$, if $A \cap \Gamma$ contains a basis of the restriction of N to the face determined by Γ)
- $S = \mathbb{Z}_{\geq 0}A$ is the semigroup generated by A. If S/Γ denotes the image semigroup of S in the quotient free group $N_{\mathbb{R}}/\mathbb{R}\Gamma$, with $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. then

$$u(\Gamma) := \mathrm{vol}(\mathrm{conv}\big(S/\Gamma) \setminus \mathrm{conv}(S/\Gamma \setminus \{0\})\big),$$



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- $Y := \operatorname{Spec}\mathbb{C}[K^\vee \cap N^\vee]$ is toric affine with Gorenstein singularities (due to the hyperplane condition)
- Any regular triangulation of the polytope Q with vertices among the elements of $\mathcal{A}$ induces a simplicial fan Σ and the associated DM Calabi–Yau stack X_Σ and a natural crepant birational morphism $\pi : X_\Sigma \to Y$.

Different triangulations give rise to different toric birational models X_Σ for the crepant resolution of the toric affine Gorenstein singularity Y.

The secondary polytope

The secondary polytope $S(A)$ is the convex hull in $\mathbb{R}^A = \mathbb{R}^n$ of the characteristics functions ϕ_Σ for all the simplicial fans Σ with

$$\phi_\Sigma(v) := \sum_{v \in \mathrm{Vert}(\sigma)} \mathrm{vol}(\sigma),$$

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The principal A–determinant E_A and the secondary polytope $S(A)$ are related in a remarkable way as shown by GKZ.

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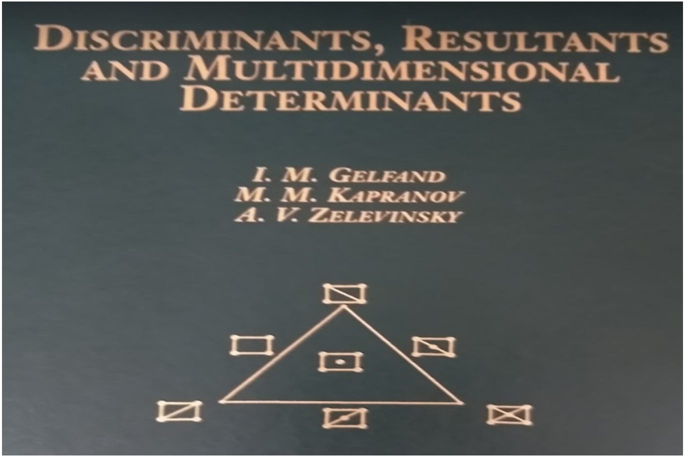
$S(A)$ (or its dual secondary fan) provides a toric compactification of the moduli stack of complex structures $\mathcal{M}_{cplex}(f)$.


DISCRIMINANTS, RESULTANTS AND MULTIDIMENSIONAL DETERMINANTS
I. M. GELFAND
M. M. KAPRANOV
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Edges and circuits

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- A *circuit* in A is a minimal dependent subset $\{v_i, i \in I\}$ with $I \subset \{1, \ldots, N\}$. In particular any circuit determines a relation of the form

$$\sum_{i \in I_+} l_i v_i + \sum_{i \in I_-} l_i v_i = 0,$$

with $I = I_+ \cup I_-$, where the two subsets $I_+ := \{i : l_i > 0\}$ and $I_- := \{i : l_i < 0\}$ are uniquely defined by the circuit up to replacing I_+ by I_-.

The topological mirror symmetry map

Let $\beta \in N$ be a lattice element.

The *bbGKZ hypergeometric system* (Borisov-H (2013)) is a system of PDE's in $\Phi_v(z_1, \ldots, z_n)$ for all $v \in \mathcal{C} \cap N$, $\mathcal{C} = \sum \mathbb{R}_{\geq 0} v_i$:

$$\frac{\partial}{\partial z_i} \Phi_v = \Phi_{v_i + v}, v \in \mathcal{C} \cap N, 1 \leq i \leq n,$$

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It can be defined even if N has torsion.

- *Gelfand, Kapranov and Zelevinsky* (GKZ) showed that this system is holonomic with regular singular points, so the number of solutions at a generic point is finite. For the bbGKZ the generic rank is vol(Q) even when it is not the case for the classical GKZ.

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- GKZ computed the corresponding characteristic cycle in $(\mathbb{C}^n)^\vee \times \mathbb{C}^n$. It has the form

$$\sum_{\Gamma \subset Q} m_\Gamma T^*_{X_\Gamma}(\mathbb{C}^n)^\vee$$

where X_Γ is a toric subvariety in $(\mathbb{C}^n)^\vee$ determined by the face Γ, and m_Γ are combinatorially defined.

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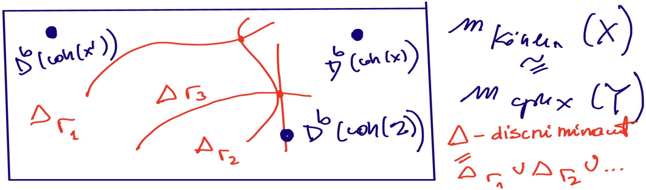
- There exists an open domain $U_\Sigma \subset \mathbb{C}^n$ and a topological mirror symmetry map

$$MS_\Sigma : (K_0(X_\Sigma) \otimes \mathbb{C})^\vee \to Sol(U_\Sigma),$$

for any stacky fan Σ. The map is compatible with the action of Fourier–Mukai functors on $D^b(coh(X_\Sigma))$ and the monodromy action on the GKZ solutions obtained by analytic continuation (*Borisov-H. (2006)*).

Categorical Statements

Intuitive picture (compatible with HMS) of the moduli spaces (stringy Kähler and mirror complex, respectively) with a compactification given by the secondary polytope $S(A)$ in the toric case.



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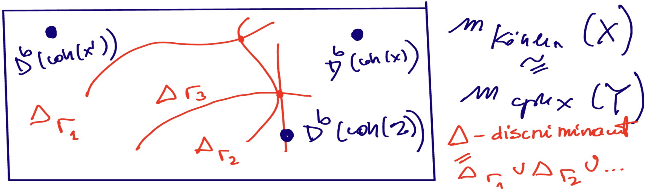

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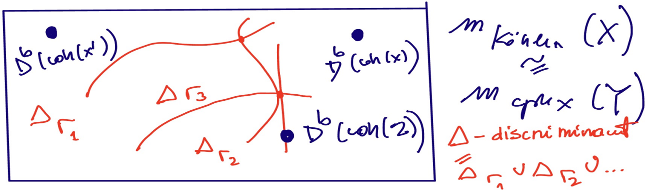

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The Aspinwall-Plesser-Wang conjecture (2016) is a proposal about the categorical picture along the components of the discriminant. It is reminiscent of the (Φ, Ψ) "vanishing–nearby cycle" construction in singularity theory.

Spherical functors

The “wall monodromy” spherical functor.

Theorem

For any edge F of the secondary polytope $S(A)$, there exists a toric DM stack Z_F and an EZ–spherical wall–monodromy functor $D^b(Z_F) \to D^b(X)$ where X is the toric DM stack induced by either one of the simplicial fans corresponding to the edge F.

Let Γ be a non-empty face of the polytope $Q = \mathrm{conv}(A)$ and Σ a stacky fan supported on K.

The stacky fan Σ_Γ is induced by the canonical projection $\pi : N \to N/\mathbb{Z}(A \cap \Gamma)$. The one dimensional cones of this stacky fan Σ_Γ are independent of Σ but the cones of the cones of the induced fan Σ_Γ are not.

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Theorem

- *For any two choices of stacky fans Σ_Γ and Σ'_Γ as above, the bounded derived categories of coherent categories $D^b(coh(X_{\Sigma_\Gamma}))$ and $D^b(coh(X_{\Sigma'_\Gamma}))$ are equivalent.*
- $\mathrm{rk}\ K_0(X_{\Sigma_\Gamma}) = u(\Gamma) \cdot i(\Gamma)$.

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Set $D^b(Z_\Gamma) := D^b(coh(X_{\Sigma_\Gamma}))$

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1) *(Aspinwall–Plesser–Wang) For each face* Γ*, there exist spherical functors* $D^b(Z_\Gamma) \to D^b(X)$ *for any toric DM stack* X *determined by a triangulation corresponding to a vertex of the secondary polytope.*

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The first part is a direct consequence of the second: the wall monodromy functors $D^b(Z_F) \to D^b(X)$ are spherical, so a result of Kuznetsov and Halpern-Leistner–Shipman implies that each component of the semiorthogonal decomposition determines a spherical functor.

Theorem (H.–Katzarkov)

For any edge F of the secondary polytope $S(A)$, the following equality holds:

$$\mathrm{rk}(K_0(D^b(Z_F))) = \sum_{\Gamma \subset Q} n_{\Gamma,F} \cdot \mathrm{rk}(D^b(K_0(Z_\Gamma))),$$

for some combinatorially defined non-negative integer multiplicities $n_{\Gamma,F}$.

Asymptotic Multiplicites

The proof is based on an analysis of the asymptotic properties of the A–determinant E_A in a limit corresponding to the edge F of the secondary polytope. Each edge F determines a circuit I, and a discriminant Δ_I. The asymptotic expansions are expressed as powers of Δ_I.

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The leading asymptotic term of E_A is $\Delta_I^{\mathrm{rk}(K_0(D^b(Z_F)))}$.

An example

- X is the resolution of the A_3 singularity, with $v_0 = (1,0), v_1 = (1,1), v_2 = (1,2), v_3 = (1,3), v_4 = (1,4)$.

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- The spherical functor is

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- The the leading term with respect to the edge F_2 in the quartic discriminant Δ_Q is

$$256a_0^3a_4^3 - 128a_0^2a_2^2a_4^2 + 16a_0a_2^4a_4 = 16a_0a_4 \cdot \Delta_I^2.$$

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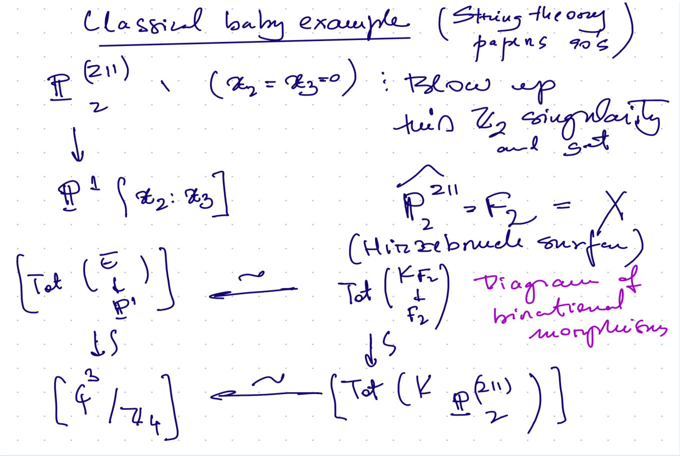
- The circuit relation I is $v_0 - 2v_2 + v_4 = 0$. The discriminant Δ_I is $4a_0a_4 - a_2^2$.
- The the leading term with respect to the edge F_2 in the quartic discriminant Δ_Q is

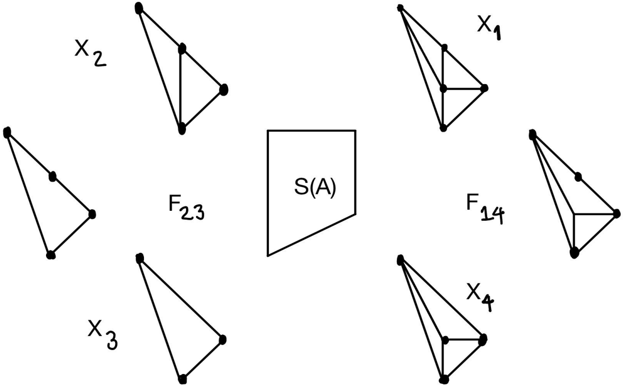
$$256a_0^3a_4^3 - 128a_0^2a_2^2a_4^2 + 16a_0a_2^4a_4 = 16a_0a_4 \cdot \Delta_I^2.$$

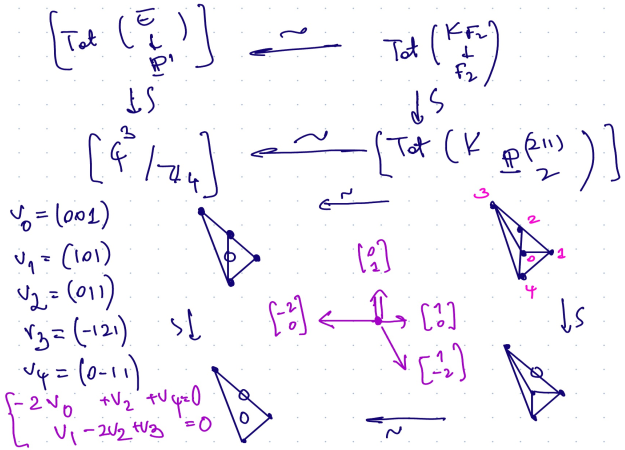
- $n_{Q,F_2} = 2$ and the spherical functor is

$$D^b([\mathrm{Spec}\,\mathbb{C}/\mathbb{Z}_2]) \to D^b([\mathbb{C}^2/\mathbb{Z}_4]).$$

Another example




X2
X1
F23
S(A)
F14
X3
X4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$v_0 = (001)$
$v_1 = (101)$
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$v_3 = (-121)$
$v_4 = (0-11)$
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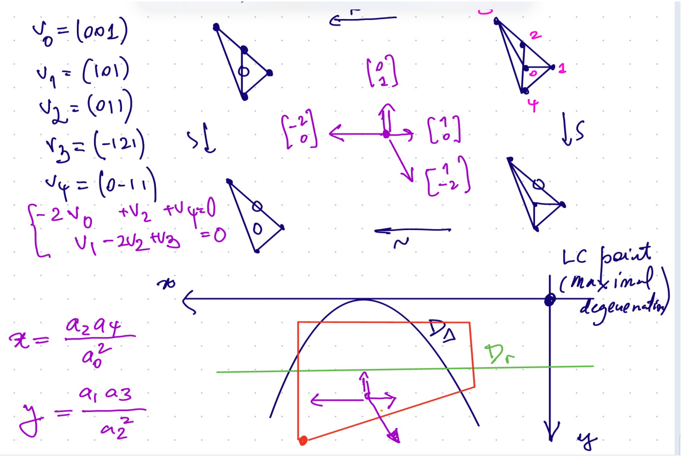
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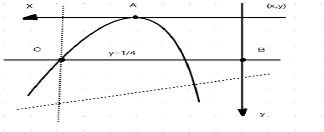
$y = \frac{a_1 a_3}{a_2^2}$
LC point (maximal degeneration)
D_Δ
D_r

Wall monodromies around the component $y = 1/4$



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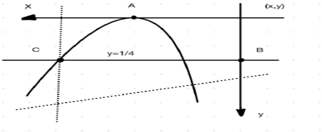
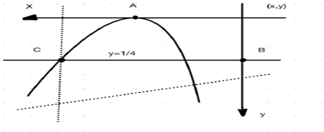


$x = k - -\text{constant}$, k very small.

$$\begin{array}{ccc} E = \mathbb{A}^1 \times \mathbb{P}^1 & \hookrightarrow & X_1 \\ \downarrow q & & \\ Z = \mathbb{A}^1 & & \end{array}$$

Wall monodromies around the component $y = 1/4$



$x = k - -\text{constant}$, k very small.

$$\begin{array}{ccc} E = \mathbb{A}^1 \times \mathbb{P}^1 & \hookrightarrow & X_1 \\ \downarrow q & & \\ Z = \mathbb{A}^1 & & \end{array}$$

$x = k - -\text{constant}$, k very large.

$$\begin{array}{ccc} E = [\mathbb{A}^1/\mathbb{Z}_2] \times \mathbb{P}^1 & \hookrightarrow & X_3 = [\mathbb{C}^3/\mathbb{Z}_4] \\ \downarrow q & & \\ Z = [\mathbb{A}^1/\mathbb{Z}_2] & & \end{array}$$

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Thank you for your attention!