LECTURE NOTES - MATH 58J (SPRING 2022)

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Contents

1. Feb 24, 2022: Introduction, $0th$ and $1st$ Cohomology of Open Subsets of Euclidean Space (ad-hoc Definitions)

Consider the n dimensional Euclidean space

$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}, i \in [n]\},$$

where $[n] \coloneqq \{1, \dots, n\}$. Abusing notation x_i's will denote coordinate values of points but also coordinate functions.

$\mathbb{R}^n$ has a metric given by

$$d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

This induces a topology on $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be an open subset. Note that this can be a complicated space.

Today and the next lecture, we will discuss differential 0 and 1 forms on U and see how these can be used to analyze the topology of U.

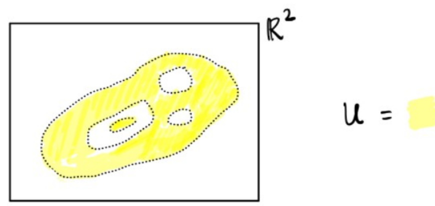


Figure 1. An example of an open subset of Euclidean space

Before that, some general remarks:

- In this class, we will measure the complexity of the topology of U (or more generally manifolds) using singular homology and cohomology. We don't know anything about these yet. Today we will give some ad-hoc definitions but the general discussion will start in the third week.
- There is a widely accepted definition of the singular cohomology of a topological space, but there are many, drastically different ways of computing it for smooth manifolds. Our class is about using differential forms to do this: deRham theory.
- Up to many technical details, you can intuitively think about a degree k cohomology class β on U as a way of associating a real number $\beta(Z)$ to each compact, boundariless, not necessarily

connected[1], oriented submanifold[2] Z of dimension k such that the following condition $(\star)$ holds.

$$\boxed{\text{If } Z \text{ is the oriented boundary of a } (k+1)-\text{dimensional submanifold with boundary, then } \beta(Z)=0.}$$

Let's refer to such Z as "$k-$cycles" - in quotation marks because we will use this word with a different meaning later

- The main operation that one does with a differential $k-$form is to integrate them along $k-$dimensional oriented submanifolds and we use this to associate real numbers to "$k-$cycles".
- Property $(\star)$ will only hold if the differential form is closed.

1.1. **Cohomology of $U \subset \mathbb{R}^n$.** Let $\pi_0(U)$ be the set of all connected components of U.

Definition 1. $H^0(U, \mathbb{R})$ is defined as the vector space of all maps from $\pi_0(U)$ to $\mathbb{R}$. □

Let $b \in U$ and $\pi_1(U, b)$ be the fundamental group of U with base point b. Recall that

$$\pi_1(U,b) := \frac{\{(S^1,*) \to (U,b) \text{ continuous}\}}{\text{homotopy preserving the base points}},$$

where $S^1 = \frac{[0,1]}{0\sim 1}$ and $* = [0] \in S^1$.
Here are some properties

- $\pi_1(U, b)$ is a group.
- Choosing a continuous path $\gamma : [0, 1] \to U$ from b to b' gives rise to a group isomorphism $f_\gamma : \pi_1(U, b) \to \pi_1(U, b')$.

Definition 2. Assuming that U is connected we define $H^1(U, \mathbb{R})_b$ as the vector space of group homomorphisms

$$\pi_1(U, b) \to \mathbb{R}.$$

□

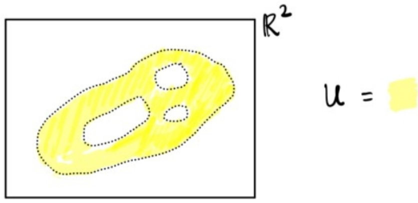
Exercise 1. Prove that for any $b, b' \in U$, as long as U is connected, there is a canonical isomorphism $H^1(U, \mathbb{R})_b \to H^1(U, \mathbb{R})_{b'}$. □

As a consequence of this exercise we can write $H^1(U, \mathbb{R})$ without any ambiguity.

[1] I write the last two only to stress the point

[2] submanifold here means a subset that locally looks like a k dimensional Euclidean space

Example 1. Let U be defined as below.



Then, $\dim(H^1(U, \mathbb{R})) = 3$. □

2. Feb 24, 2022: Degree 0 and 1 Differential Forms on Open Subsets of Euclidean Space, a Special Case of de Rham Theorem

- A differential 0-form on U is a smooth[3] function $U \to \mathbb{R}$.
- A differential 0-form f is called closed if $\frac{\partial f}{\partial x_i} \equiv 0, \forall i \in [n]$

Proposition 1. *There is a canonical linear isomorphism*

$$H^0_{dR}(U) := \{\textit{closed differential 0-form on } U\} \simeq H^0(U; \mathbb{R})$$

Remark 1. In general $H^k_{dR} := \frac{\{\text{closed differential k-form on U}\}}{\{\text{exact differential k-form on U}\}}$ □

- A differential 1-form on U is an expression $f_1 dx_1 + ... + f_n dx_n$ where $f_i : U \to \mathbb{R}$ are smooth functions
- A differential 1-form $\alpha = \sum_{i=1}^{n} f_i dx_i$ is called exact, if for some smooth $V : U \to \mathbb{R}$,
$$f_i = \frac{\partial V}{\partial x_i}, \forall i \in [n].$$
In this case we write $\alpha = dV$.
- A differential 1-form is closed if for all $i \neq j \in [n]$,
$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0.$$

Lemma 1. *If $\alpha = \sum_{i=1}^{n} f_i dx_i$ is exact, then it is closed.*

Proof. Since it is exact, $\exists V : U \to \mathbb{R}$, such that $f_i = \frac{\partial V}{\partial x_i}$, so

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} = \frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial f_j}{\partial x_i}.$$

[3]note that this is a condition much stronger than differentiable, it means that all iterated partial derivatives exist. please read the wikipedia page if you are not familiar.

$\square$

Exercise 2. For $n = 2$ and $n = 3$ explain what it means for the differential 1-form $\alpha = \sum_{i=1}^{n} f_i dx_i$ to be closed in terms of the vector field $F = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ using terms from your calculus classes. Recall Green's and Stokes theorems. $\square$

Theorem 1. *Assuming that U is connected, there exists a linear isomorphism,*

$$H^1_{dR} := \frac{\{\textit{closed differential 1-forms on } U\}}{\{\textit{exact differential 1-forms on } U\}} \simeq H^1(U;\mathbb{R}) \tag{1}$$

Proof sketch. First we want to define a linear map

$$\int : \{\text{closed 1-forms}\} \to \{\pi_1(U,b) \to \mathbb{R} \quad \text{group homomorphisms}\}$$

Recall: $X \subset \mathbb{R}^n$ arbitrary subset. A map $g : X \to \mathbb{R}^m$ is called smooth if it extends to a smooth map $N(X) \to \mathbb{R}^m$ where $N(X)$ is an open neighbourhood of X.

Fact:

- Any class in $\pi_1(U,b)$ can be represented by a smooth map $(S^1, *) \to (U,b)$.
- Any two smooth maps $S^1 \to U$ that are homotopic continuously are homotopic smoothly.

Recall: Given $\alpha = \sum_{i=1}^{n} f_i dx_i$ and a smooth path $\gamma : [0,1] \to U$, we can define the line integral $\int_\gamma \alpha := \int_0^1 F \cdot \gamma' dt$, where $F = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$.

- The map $\int$ is independent of the parametrization of γ, meaning, if $\phi : [0,1] \to [0,1]$ is a smooth bijective map with $\phi' \neq 0$, then $\int_\gamma \alpha = \int_{\gamma\circ\phi} \alpha$. So line integral only depends on the image of γ.

Back to the map $\int$: we send a given $\alpha \in \{\text{closed 1-forms}\}$ to the map $\int_\cdot \alpha : \pi_1(U,b) \to \mathbb{R}$ defined by

$$\alpha \mapsto \int_{\bar{\gamma}} \alpha,$$

where $\bar{\gamma}$ is an arbitrary smooth representative of a. In the exercise below you will show that this is well defined. Assuming that for now, it is easy to see that $\int_\cdot \alpha$ is a group homomorphism[4], so an element of $H^1(U;\mathbb{R})$, and the resulting $\int$ is a linear map.

[4] I forgot to say this in class, so please check it for yourself!

Exercise 3. Show that the map $\int$ is well defined by proving if $\alpha = \sum_{i=1}^{n} f_i dx_i$ is closed and $\gamma, \gamma' : S^1 \to U$ are smooth maps that are smoothly homotopic, then $\int_\gamma \alpha = \int_{\gamma'} \alpha$. (**Hint:** Start by analyzing $n = 2, 3$ and where the smooth homotopy $S^1 \times [0,1] \to U$ is injective, then reduce the statement to Green's theorem. Additionally, you may want to check proof of Stokes theorem.)

$\square$

FIGURE 2. An example of an injective smooth homotopy's image for $n = 2$.

Goals:

(1) Prove that $\int$ sends an exact differential 1-form to zero. We obtain a linear map $\widetilde{\int} : H^1_{dR}(U, \mathbb{R}) \to H^1(U, \mathbb{R})$.
(2) Prove that $\widetilde{\int}$ is injective. ("Construct a potential")
(3) Prove that $\widetilde{\int}$ is surjective.

We start with 1). Let us integrate $\alpha = df$ along a smooth loop γ.

$$\int_\gamma \alpha = \int_0^1 \nabla f \cdot \gamma'(t) dt \overset{\text{FTC}}{=} f(b) - f(b) = 0.$$

Denote the resulting map by

$$\widetilde{\int} : \frac{\text{Closed differential 1-forms}}{\text{Exact differential 1-forms}} \to H^1(U, \mathbb{R}).$$

For 2), we need to show that if $\int_\gamma \alpha = 0 \quad$ for all $\quad \gamma : (S^1, \star) \to (U, b)$, then $\alpha = df$ for some $f : U \to \mathbb{R}$.

Exercise 4. Do this! This is the same task as constructing a potential (recall work integrals, conservative fields etc.) for the corresponding vector field. $\square$

Let us make a simplifying assumption on U to not deal with orthogonal difficulties in 3). Assume there exists $\gamma_1, \gamma_2, .., \gamma_n \in \pi_1(U, b)$ that freely generates the abelianization of $\pi_1(U, b)$. This implies that giving a group homomorphism $\pi_1(U, b) \to \mathbb{R}$ is equivalent to assigning real numbers to each of $\gamma_1, \gamma_2, .., \gamma_n$ (arbitrarily).

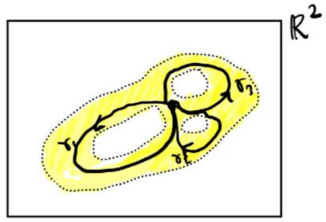


FIGURE 3

Now we need to create a differential 1-form that integrates to any $a_1, a_2, .., a_n \in \mathbb{R}$ along $\gamma_1, \gamma_2, .., \gamma_n$. This is still quite difficult. That will follow from DeRham Theorem, which will be the highlight of our course.

Remark 2. Once we give the general definition of singular cohomology, I will assign a homework exercise which shows that it agrees with what we defined today in degrees 0 and 1. The analogous statement will be automatic for DeRham cohomology. □

Exercise 5. Finish the proof of surjectivity in the case

$$U = \mathbb{R}^2 - \text{finitely many points.}$$

(**Hint:**Start with $\mathbb{R}^2 - (0, 0)$ and use the closed differential 1-form $\alpha = -\frac{ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}$.) □

3. March 03, 2022: Manifolds, Charts, Smooth Atlases

Riemann was looking for a class of spaces which exist by themselves, (for example, they don't have to be embedded in an Euclidean space $\mathbb{R}^N$) with the following properties[5]. Let X be such a space:

- X admits local coordinates. This means that the points x sufficiently near any $x_0 \in X$ are determined uniquely by the values of a set of real valued coordinates $x_1, x_2, \cdots, x_n$:
$$x = (x_1, \ldots, x_n).$$
This is sometimes called a generalized coordinate system in physics. There could be many such generalized coordinate systems near a given point. It is important that often generalized coordinates do not extend to the entirety of X.

[5]particularly vague phrases are underlined

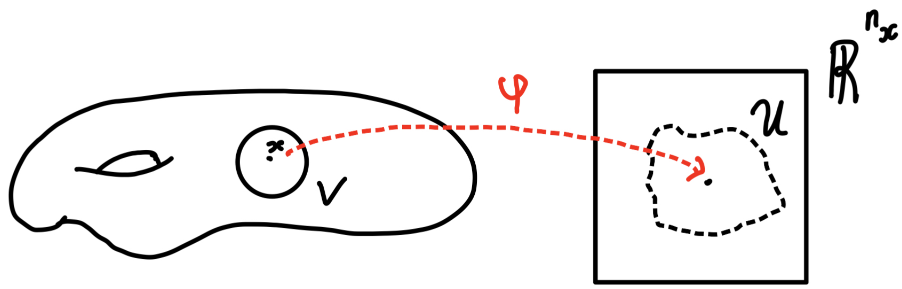


FIGURE 4

- One can use techniques of calculus. This means, in particular, that there should be a large class of $C^1/C^2/\cdots/$ smooth functions $X \to \mathbb{R}$. If two generalized coordinate systems are related to each other by a non-differentiable transformation, then a function $X \to \mathbb{R}$ that is differentiable with respect to one may not be differentiable with respect to the other.

Definition 3. A topological space X is called a topological manifold if for every $x \in X$, there exists a nonnegative integer $n_x \geq 0$, an open subset $U \subset \mathbb{R}^{n_x}$, an open neighbourhood $V \subset X$ of x and a homeomorphism $\phi : V \to U$ (See Figure 4). □

Remark 3. Note that being a topological manifold is a property. Usually in this definition one also assumes that X is Hausdorff and, less often but still quite often, second countable. We will focus on the core part of the definition today. Later on we will start assuming these two properties when they are needed. □

Definition 4. Let X be a topological space. Let us call $U \subset \mathbb{R}^n$, $V \subset X$ open and $\phi : V \to U$ homeomorphism a coordinate chart in X. V is called the domain of the chart and the functions $x_1, \cdots, x_n$ obtained by $x_i : V \to U \xrightarrow{pr_i} \mathbb{R}$ the coordinates of the chart. □

Fact (A consequence of Invariance of Domain)
If an open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$, then $m = n$.

Exercise 6. Using the fact above, prove that n_x in the definition is uniquely determined. Also, prove that $X \to \mathbb{Z}_{\geq 0}, x \mapsto n_x$ is constant on connected components of X. □

Definition 5. If $n_x = n$ for all $x \in X$, then we say that X is $n-$dimensional. We write this briefly by X^n. □

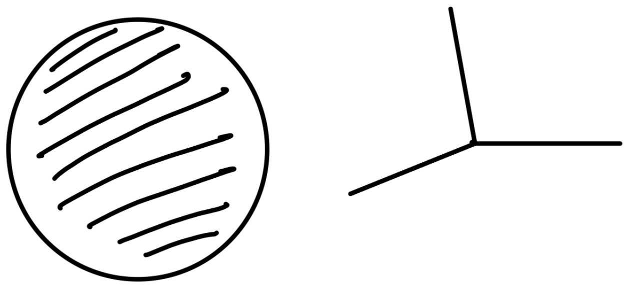


FIGURE 5. Non-Examples of Manifolds

From now on, when we say X is a topological manifold, we assume that there is such an $n \geq 0$.

Example 2. Topological Manifolds

- $\mathbb{R}^n$
- $S^n : \{x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$.

□

To see that S^n is a topological manifold, we can use the stereographic projection. Let us consider a point $y_0 \in S^n$, and let H_0 be the hyperplane that is tangent to the point opposite to $y_0 \in S^n$ (we call this point $-y_0$). For every, $y \in S^n \setminus \{y_0\}$ the straight line l_y passing through y and y_0 intersects H_0 at precisely one point.

$$S_{y_0} : S^n \setminus \{y_0\} \longrightarrow H_0 \simeq \mathbb{R}^n$$
$$y \longmapsto l_y \cap H_0$$

Note that $H_0 \cong$Parallel hyperplane passing through the origin$\cong \mathbb{R}^n$. The second homeomorphism can be obtained by choosing a basis.

Proposition 2. *S_{y_0} is a homeomorphism.*

Proof. (sketch)
$n = 0$ case is given in the figure. In this case the map is identity.

Exercise 7. Do the $n = 1$ case. □

We can deduce the $n > 1$ case by using rotational symmetry around the line that contains the diameter (passing through y_0 that is perpendicular to H_0). The stereographic projection in dimension n is given by spinning around l_y the stereographic projection in dimension $n-1$. □

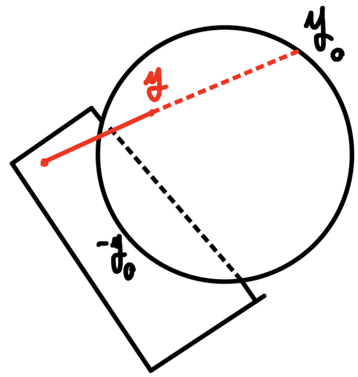


FIGURE 6. Stereographic Projection

FIGURE 7. $n = 0$ case

Remark 4. Stereographic projection

- Preserves angles.
- It preserves circles ($n = 2$).
- But, it distorts distances.

□

Exercise 8. Prove that $\{x^2 - y^3 = 1\} \subset \mathbb{C}^2$ is a topological manifold. (**Hint:** Use projections to x and y.) □

Definition 6. Let X be a topological space, and $\phi_1 : V_1 \to U_1$ and $\phi_2 : V_2 \to U_2$ be coordinate charts. Then, the map $\phi_2 \circ \phi_1^{-1} : \phi_1(V_1 \cap V_2) \to \phi_2(V_1 \cap V_2)$ is called the transition map form the chart ϕ_1 to the chart ϕ_2. Note that transition maps are automatically homeomorphisms. □

Definition 7. A smooth atlas on a topological space X is a collection of charts $\{\phi_i : V_i \to U_i\}_{i \in I}$ such that
1) $\bigcup_{i \in I} V_i = X$
2) The transition map from any chart in the collection to any other in the collection is smooth. □

Remark 5. Atlas means a book of maps, i.e. images of charts $\phi : V \to U \subset \mathbb{R}^2$ on the manifold that is the surface of the earth. It is likely that some of these maps are drawn using stereographic projection. □

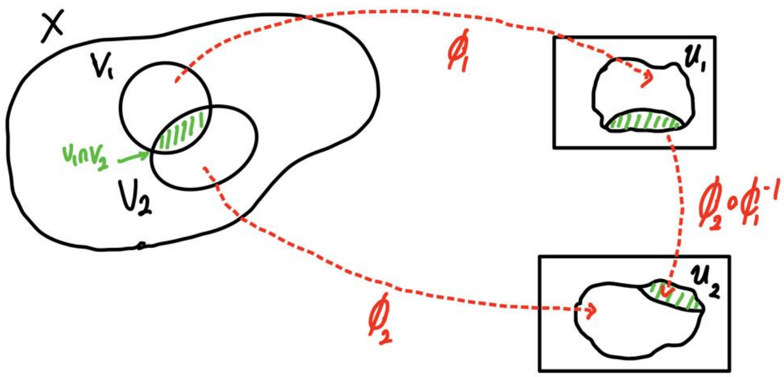


FIGURE 8. Transition map between charts

Exercise 9. Using latitude and longitude, define a chart on S^2 with domain $S^2 \backslash 0^{th}-$meridian and prove that it has smooth transition maps to all stereographic projections. You can assume that stereographic projection charts form a smooth atlas. □

4. March 07, 2022: Definition of Singular Homology

Convention: Unless otherwise stated, all of our vector spaces and chain complexes (to be defined) are over $\mathbb{R}$.

Definition 8. A graded vector space V_* is a collection of vector spaces $\{V_i\}_{i\in\mathbb{Z}}$ indexed by $\mathbb{Z}$. V_* is non-negatively graded if $V_i = 0$ for $i < 0$.[6] □

Definition 9. A chain complex (C_*, ∂_*) is a graded vector space C_* with a collection of linear maps $\partial_n : C_n \to C_{n-1}$, $n \in \mathbb{Z}$, such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. We call ∂_n's boundary maps. □

$$\ldots \leftarrow C_{-2} \xleftarrow{\partial_{-1}} C_{-1} \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \leftarrow \ldots$$

Definition 10. The homology of a chain complex (C_*, ∂_*) is a graded vector space $H_*(C_*, \partial_*)$ defined by

$$H_n\left((C_*, \partial_*)\right) := \frac{\operatorname{Ker}(\partial_n : C_n \to C_{n-1})}{\operatorname{Im}(\partial_{n+1} : C_{n+1} \to C_n)}$$

□

It immediately follows from $\partial_i \circ \partial_{i+1} = 0$ that $\operatorname{Im}(\partial_{i+1}) \subset \operatorname{Ker}(\partial_i)$. There is a slight variant of the last two definitions.

[6] V=0 stands for the trivial vector space with 0 as the only element, $V = \{0\}$.

Definition 11. C^* graded vector space with $d_n : C^n \to C^{n+1}$ coboundary maps such that $d_n \circ d_{n-1} = 0$. (C^*, d_*) is called a co-chain complex.

$$H^n\left((C^*, d_*)\right) := \frac{\mathrm{Ker}(d_n)}{\mathrm{Im}(d_{n-1})}$$

is called cohomology. □

$$\ldots \to C^{-2} \xrightarrow{d_{-2}} C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_{-1}} C^2 \to \ldots$$

Now we move on to define the singular chain complex $C_*(X;\mathbb{R})$ of a topological manifold X.

Definition 12. The n-dimensional simplex Δ^n for $n \geq 0$ is defined as

$$\Delta^n := \left\{ (x_0, \ldots, x_n) \;\middle|\; \begin{array}{l} x_i \geq 0, \quad \forall i = 0, \ldots, n \\ x_0 + \ldots + x_n = 1 \end{array} \right\}$$

□

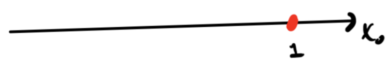


FIGURE 9. 0-dimensional simplex.

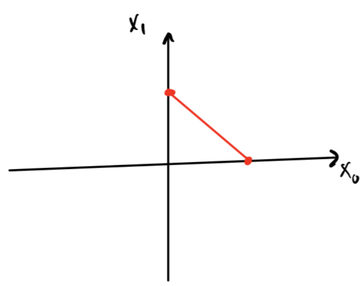


FIGURE 10. 1-dimensional simplex.

For each subset $S \subset \{0, 1, \ldots, n\}$, we can define a subset (a face) by

$$F_S := \left\{ (x_0, \ldots, x_n) \;\middle|\; \begin{array}{l} x_i = 0, \quad \text{if } i \in \{0, \ldots, n\} \setminus S \\ (x_0, \ldots, x_n) \in \Delta^n \end{array} \right\}$$

As an example, $F_{\{i\}}$, $i = 0, \ldots, n$ correspond to vertices.

Exercise 10.

- Prove that the dimension of F_S is $|S| - 1$. Explain what you mean by dimension.

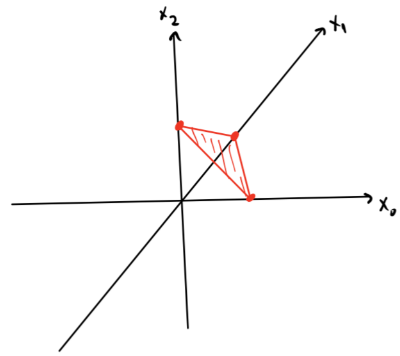


FIGURE 11. 2-dimensional simplex.

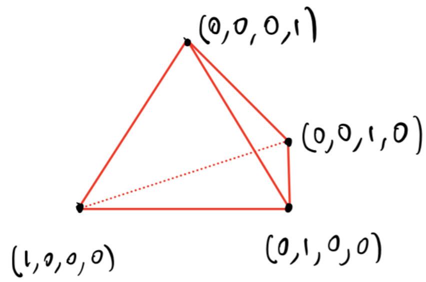


FIGURE 12. 3-dimensional simplex.

- Prove that $F_{S_1} \cap F_{S_2} = F_{S_1 \cap S_2}$.

$\square$

Definition 13. For each $n \geq 1$ and $0 \leq i \leq n$ we define the face map $f_{i,n} : \Delta^{n-1} \to \Delta^n$ with $(x_0, ..., x_{n-1}) \mapsto (y_0, ..., y_n)$ where

$$y_j = \begin{cases} x_j \ , & j < i \\ 0 \ , & j = i \\ x_{j-1} \ , & j > i \end{cases}$$

$\square$

- This simply adds a zero to the $(i+1)$th slot.
- The image of $f_{i,n}$ is $F_{\{0,...,n\} \setminus \{i\}}$

We need one last notion before we define the singular chain complex.

Definition 14. Given any set A, we define the vector space generated by A as the vector space of all finite formal linear combinations of the elements of A.

$$\left\{ \sum_{a \in A} c_a \cdot a \middle| c_a \in \mathbb{R} \text{ and } c_a \neq 0 \text{ for finitely many elements} \right\}.$$

$\square$

Exercise 11. Construct a natural linear map from the vector space generated by A to the vector space of all maps $A \to \mathbb{R}$. Prove that this map is an isomorphism if and only if A is finite. Bonus: analyze when these two vector spaces are isomorphic - by an arbitrary map. $\square$

Consider the subspace topology on simplices.

4.1. **Singular Homology of a Topological Space.** Let X be a topological space. For $n \geq 0$, $C_n(X;\mathbb{R})$ is defined to be the vector space generated by the set of all continuous maps $\Delta^n \to X$. The elements of $C_n(X;\mathbb{R})$ are called singular chains of degree n. We set $C_n(X;\mathbb{R}) = 0$ for all $n < 0$. So $C_*(X;\mathbb{R})$ is a non-negatively graded vector space. Now we will equip it with boundary maps and turn it into a chain complex. Let $n \geq 1$. For any continuous $g : \Delta^n \to X$ we define

$$\partial_n g = \sum_{i=0}^{n} (-1)^i g \circ f_{i,n} \in C_{n-1}(X;\mathbb{R})$$

where $g \circ f_{i,n} : \Delta^{n-1} \xrightarrow{f_{i,n}} \Delta^n \xrightarrow{g} X$. We then extend to all singular chains so that map is linear and we get

$$\partial_n : C_n(X;\mathbb{R}) \to C_{n-1}(X;\mathbb{R}) \text{ for } n \geq 1$$

and $\partial_n = 0$ for $n < 1$.

Example 3. As an example, consider the following figure □

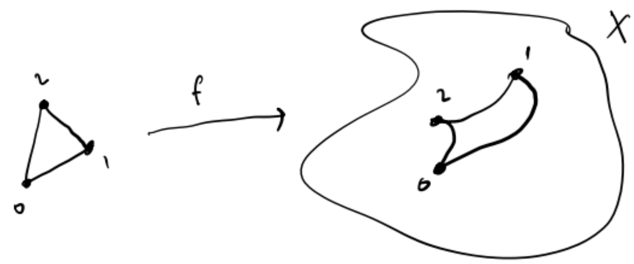


FIGURE 13. Example

Now we look at the boundary maps $\partial_2 f$ of f.

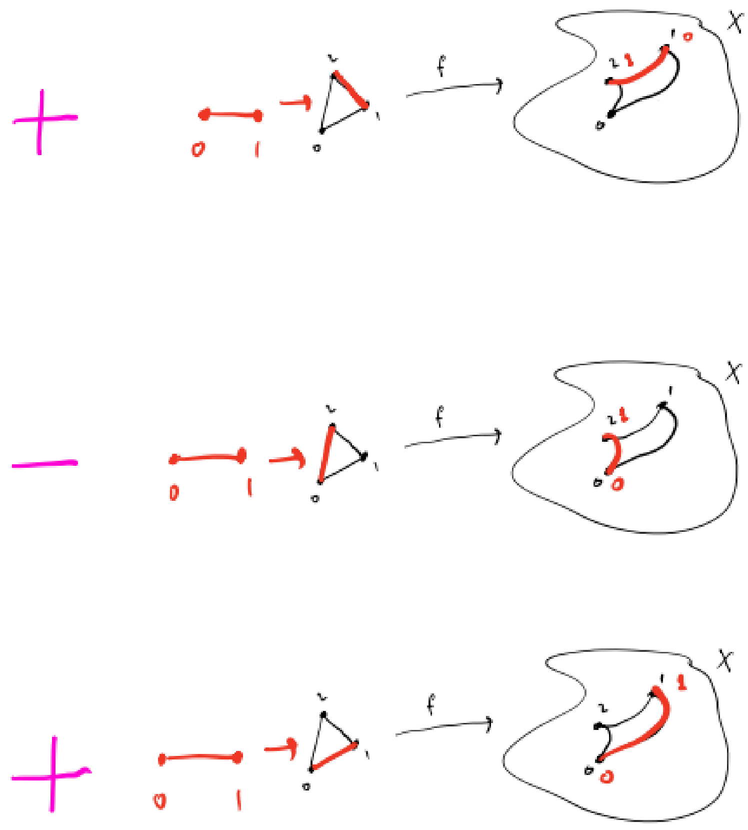


FIGURE 14. Boundary maps of the example above.

Proposition 3. $\partial_{n-1} \circ \partial_n = 0$

Proof. For $n < 2$ it is obvious. For $n \geq 2$ it suffices to show that for $g : \Delta^n \to X$ we should have $\partial_{n-1}(\partial_n(g)) = 0$.

$$\partial_{n-1}(\partial_n(g)) = \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i g \circ f_{i,n}\right) = \sum_{i=0}^{n}(-1)^i \partial_{n-1}(g \circ f_{i,n})$$
$$= \sum_{i=0}^{n}(-1)^i\left(\sum_{j=0}^{n-1}(-1)^j g \circ f_{i,n} \circ f_{j,n-1}\right) = \sum_{i,j}(-1)^{i+j} g \circ f_{i,n} \circ f_{j,n-1}$$

Where $f_{i,n} \circ f_{j,n-1} : \Delta^{n-2} \to \Delta^n$...

□

Exercise 12. Finish the proof of this proposition. If you were not able to follow in class, first do it for $n = 2$ using the pictures above - no need to write this, just to get yourself oriented. □

Definition 15. We define the singular homology of X as the homology of its singular chain complex

$$H_n(X;\mathbb{R}) \coloneqq H_n(C_*(X;\mathbb{R}), \partial_*)\ .$$

□

5. March 10, 2022: Constructing Singular Cycles, Homology of a Point, Star Shaped Open Subsets of Euclidean Space

Definition 16. Let us call the elements of

$$Z_n(X;\mathbb{R}) \coloneqq ker(\partial_n : C_n(X;\mathbb{R}) \to C_{n-1}(X;\mathbb{R}))$$

the singular n-cycles and the elements of

$$B_n(X;\mathbb{R}) \coloneqq im(\partial_{n+1} : C_{n+1}(X;\mathbb{R}) \to C_n(X;\mathbb{R}))$$

the singular n-boundaries. □

In this class, often we will omit the adjective singular from these phrases for brevity. Then, by definition

$$\begin{aligned} H_n(X;\mathbb{R}) &= \frac{Z_n(X;\mathbb{R})}{B_n(X;\mathbb{R})} \\ &= \frac{n\text{-cycles}}{n\text{-boundaries}} \end{aligned}$$

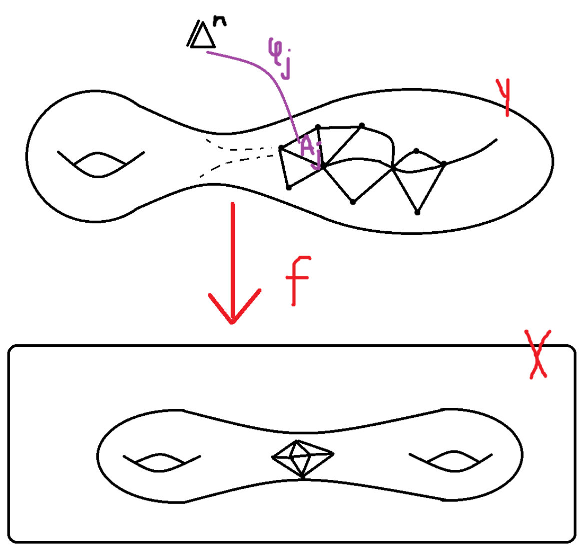
5.1. **Singular n-cycles From Geometric n-cycles - Slightly Informal Discussion.** Let X be a topological manifold. Let us define a geometric n-cycle to be the image of a continuous map $f : Y^n \to X$ where Y is a compact oriented (We will define precisely for smooth manifolds later) topological manifold.

I want to briefly explain how a geometric n-cycle gives rise to an n-cycle on X. Oriented compact submanifolds are examples of geometric n-cycles.

Under some mild conditions (for example if it is Hausdorff and admits a smooth structure), Y admits a triangulation. This, in particular, means we can find

$$Y = \bigcup_{j=1}^{N} A_j \quad \text{with homeomorphisms} \quad \phi_j : \Delta^n \to A_j$$

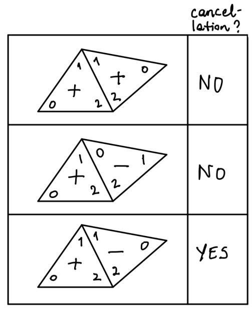
such that the intersections $A_i \cap A_j$ for $i \neq j$ are either empty or equal to the images of equi-dimensional faces under both ϕ_i and ϕ_j.



The idea then is to add up all $f \circ \phi_j : \Delta^n \to X$ and get an n-cycle seeing how the boundaries seem to cancel each other.

The issue is that we do not actually know, it depends on whether the signs work out.

We could also add $\pm f \circ \phi_j$ of course. Note that we can modify ϕ_j also by homeomorphism, $\Delta^n \to \Delta^n$ which permute the coordinates of



$\mathbb{R}^{n+1}$. If the result is $\widetilde{\phi}_j$ and the permutation π^{-1}

$$\begin{aligned}
\partial_n(f \circ \widetilde{\phi}_j) &= \sum_{i=0}^{n} (-1)^i f \circ \widetilde{\phi}_j \circ \mathrm{face}_{i,n} \\
\pi(l) = i \qquad &= \sum_{l=0}^{n} (-1)^{\pi(l)} f \circ \widetilde{\phi}_j \circ \mathrm{face}_{\pi(l),n} \\
&= \sum_{l=0}^{n} (-1)^{\pi(l)-l} (-1)^l f \circ \phi_j \circ \mathrm{face}_{l,n}
\end{aligned}$$

The interesting result is that whether one can modify ϕ_i's using these so that $\partial_n(\sum_{i=1}^{N} \pm\widetilde{\phi}_i) = 0$ is a condition that depends only on Y and is called orientability[7].

If this is true, then we obtain at least two n-cycles in Y, we can multiply everything by -1. Actually orienting Y, we would pick out one of them.

Exercise 13. Consider $S^1 \subset \mathbb{R}^2$. Construct a nonzero 1-cycle in $\mathbb{R}^2$ corresponding to this geometric 1-cycle. Prove that it is actually a 1-boundary directly. $\square$

5.2. **Some Computations.** Let X be a point. What is $H_*(X; \mathbb{R})$?

For every $n \geq 0$, there exists exactly one continuous map $\Delta^n \xrightarrow{c_n} X$. Therefore,

$$C_n(X; \mathbb{R}) = \mathbb{R} \cdot c_n.$$

[7]The first step is to choose a total order on the vertices of the triangulation. Then orientability guarantees that the signs can be chosen to achieve the necessary cancellations.

How about the boundary map?

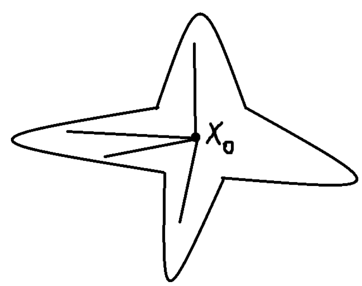
$$\begin{aligned}\partial_n c_n &= \sum_{i=0}^{n} (-1)^i c_{n-1} \\ &= \begin{cases} c_{n-1} & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}\end{aligned}$$

The singular chain complex looks like

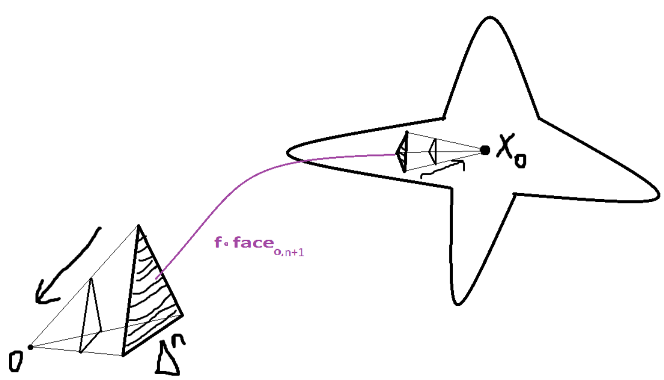
$$\leftarrow 0 \leftarrow 0 \leftarrow \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{id} \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{id} \ldots$$

$$\Longrightarrow \qquad H_0(X;\mathbb{R}) = \mathbb{R} \text{ and } H_n(X;\mathbb{R}) = 0, \;\; n \neq 0.$$

Let us now consider $U \subset \mathbb{R}^n$ open and star-shaped, that is, there exists an $x_0 \in U$ such that the line segment x_0 and y lies inside U for all $y \in U$.



We claim that $H_*(U;\mathbb{R}) \cong H_*(\text{point};\mathbb{R})$. The idea is that for a given $f : \Delta^n \to U$, we can define a $P_f : \Delta^{n+1} \to U$ as: follows



Exercise 14. Write down an explicit formula for P_f in terms of f. Extending $f \mapsto P_f$ linearly, define a linear map $P : C_n(X;\mathbb{R}) \to C_{n+1}(X;\mathbb{R})$. For $n > 0$, $\sigma \in C_n(X;\mathbb{R})$, prove that

$$\partial_{n+1} P\sigma = -P(\partial_n \sigma) + \sigma.$$

For $n = 0$, $\sigma \in C_0(X;\mathbb{R})$, what is $\partial_1 P\sigma$? □

If $\sigma \in Z_n(U;\mathbb{R})$ and $n > 0$, then $\partial_{n+1} P\sigma = \sigma$. This implies $\sigma \in B_n(U;\mathbb{R})$ and hence $H_n(U;\mathbb{R}) = 0$ for $n \neq 0$.

Exercise 15. Let Y be a topological space. Prove that $H_0(Y;\mathbb{R})$ is isomorphic to the vector space generated by the set of connected components of Y. □

6. March 14, 2022: Induced Maps on Homology, Homeomorphism Invariance, Homotopy Invariance of Induced Maps

Definition 17. Let $(C_\bullet, \partial_\bullet)$ and $(\tilde{C}_\bullet, \tilde{\partial}_\bullet)$ be chain complexes. A chain map is a collection of linear maps $C_n \xrightarrow{f_n} \tilde{C}_n$, $\forall n \in \mathbb{Z}$ such that each square in the diagram

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l] \arrow[d, "f_{n-1}"] & C_n \arrow[l, "\partial_n"'] \arrow[d, "f_n"] & \cdots \arrow[l] \\
\cdots & \tilde{C}_{n-1} \arrow[l] & \tilde{C}_n \arrow[l, "\tilde{\partial}_n"'] & \cdots \arrow[l]
\end{tikzcd}$$

is commutative, i.e.

$$f_{n-1} \circ \partial_n = \tilde{\partial}_n \circ f_n, \quad \forall n \in \mathbb{Z}.$$

□

Given a chain map $f_\bullet : C_\bullet \to \tilde{C}_\bullet$ we canonically obtain a linear map of graded vector spaces

$$H(f) : H_*(C) \to H_*(\tilde{C}).$$

Remark 6. Let us make a notational clarification. For any chain complex (C, ∂) we define

$Z_n(C) \coloneqq \ker(\partial_n) \cdot n$-cycles
$B_n(C) \coloneqq \operatorname{im}(\partial_{n+1}) \cdot n$-boundaries
$H_n(C) = {Z_n(C)}/{B_n(C)}$

When $C_\bullet = C_*(X;\mathbb{R})$ then we may add the adjective "singular". □

Definition 18. If two n-cycles in a chain complex differ by an n-boundary (that is they define the same class in homology), we say that these two cycles are homologous. $\square$

Back to constructing $H_n(C) \to H_n(\tilde{C})$.

(1) If $\sigma \in Z_n(C)$, then $f(\sigma) \in Z_n(\tilde{C})$:
$f(\partial\sigma) = \tilde{\partial} f(\sigma) \implies \tilde{\partial} f(\sigma) = 0$

(2) If $\sigma \in B_n(C)$, then $f(\sigma) \in B_n(\tilde{C})$:
$\sigma = \partial\gamma \implies f(\sigma) = f(\partial\gamma) = \tilde{\partial} f(\gamma)$

$\implies$ We obtain the map shown in dashes below.

$$\begin{tikzcd}
B_n(C) \arrow[r, hook] & Z_n(C) \arrow[r] \arrow[d] & Z_n(\tilde{C}) \arrow[r] & Z_n(\tilde{C})/B_n(\tilde{C}) \\
& Z_n(C)/B_n(C) \arrow[urr, dashed] & &
\end{tikzcd}$$

We can compose chain maps as in the following exercise.

Exercise 16. Prove that if $f : C_* \to \tilde{C}_*$ and $g : \tilde{C}_* \to \tilde{\tilde{C}}_*$ are chain maps, then $h : C_* \to \tilde{\tilde{C}}_*$ defined by $h_n := g_n \circ f_n$ is a chain map. For such f, g, h prove that $H(h) = H(g) \circ H(f)$. $\square$

Definition 19. Let $\varphi : X \to Y$ be continuous. Then for every continuous map $\rho : \Delta^n \to X$, we obtain the continuous map $\varphi \circ \rho : \Delta^n \to Y$. Linearly extending we obtain a map $(\varphi_*)_n : C_n(X;\mathbb{R}) \to C_n(Y;\mathbb{R})$. These form a chain map

$$\varphi_* : C_*(X;\mathbb{R}) \to C_*(Y;\mathbb{R}).$$

$\square$

Exercise 17. Prove that φ_* is indeed a chain map. $\square$

We also obtain

$$H\varphi_* : H_*(X;\mathbb{R}) \to H_*(Y;\mathbb{R}).$$

Exercise 18. For continuous maps $\tilde{\varphi} : X \to Y$ and $\varphi : Y \to Z$, prove that $\varphi_* \circ \tilde{\varphi}_* = (\varphi \circ \tilde{\varphi})_*$. $\square$

Corollary 1. $H\varphi_* \circ H\tilde{\varphi}_* = H(\varphi \circ \tilde{\varphi})_*$. $\square$

Since $(\mathrm{id})_*$ is the identity map, we immediately obtain that if $\varphi : X \to Y$ is a homeomorphism then $H\varphi_* : H_*(X;\mathbb{R}) \to H_*(Y;\mathbb{R})$ is a linear isomorphism.

Hence singular homology can distinguish non-homeomorphic topological spaces (it does not have to!) Actually singular homology is in general only sensitive to the homotopy equivalence class. Let's explain this.

Recall: $\bullet f, g : X \xrightarrow{cts} Y$ are called homotopic if there exists a continuous

$$F : X \times [0,1] \to Y \text{ s.t.}$$
$$F|_{\{0\}} = f \;\&\; F|_{\{1\}} = g.$$

$\bullet$ A continuous $f : X \to Y$ is called a homotopy equivalence if $\exists g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to identity maps (of Y and X).

Exercise 19. Let $U \subset \mathbb{R}^n$ be star shaped with respect to x_0. Prove the map pt. $\to U$ with image x_0 is a homotopy equivalence. □

Theorem 2. *If f & $g : X \to Y$ are homotopic, then $Hf_* = Hg_*$.*

I don't want to spend time proving this homotopy invariance theorem, but it is quite important and the proof is not too difficult. If you want we can discuss during office hours. You are responsible from the statement, not the proof. The key idea in the proof is essentially the one that we used on proving $H_*(\text{star shaped}) \cong H_*(\text{pt.})$.

The corollary below can be proved using the same logic that showed that homeomorphisms induce isomorphisms on homology but this time using Theorem 2.

Corollary 2. *If $f : X \to Y$ is a homotopy equivalence, then $Hf : H_*(X;\mathbb{R}) \to H_*(Y;\mathbb{R})$ is an isomorphism.* □

Exercise 20. Prove this corollary assuming Theorem 2. □

7. March 17, 2022: Mayer-Vietoris Property, Special Properties of Singular Homology for Manifolds, Singular Cohomology

Definition 20. An exact sequence is a sequence of vector spaces V_n, $n \in \mathbb{Z}$ and maps $V_n \xrightarrow{f_n} V_{n-1}$ such that $\forall n \in \mathbb{Z} \quad ker(f_n) = im(f_{n+1})$

$$\begin{tikzcd} V_{n-1} \arrow[r, bend right, "\mathsf{ker}"'] & V_n \arrow[l, "f_n"'] & V_{n+1} \arrow[l, "f_{n+1}"'] \arrow[l, bend left, "\mathsf{Im}"] & V_{n+2} \arrow[l, "f_{n+2}"'] \end{tikzcd}$$

$$\mathsf{ker} = \mathsf{Im}$$

□

Remark 7. This is the same data as chain complex with 0-Homology. □

Exercise 21. Let V_n be an exact sequence. Assume that $\sum \dim V_n < \infty$. Show that $\sum_{n \text{ even}} \dim V_n = \sum_{n \text{ odd}} \dim V_n$. $\square$

Theorem 3 (Mayer-Vietoris Theorem)**.** *Let X be a topological space and $U, V \subseteq X$ open subsets.*
There are canonical maps

$$H_{n+1}(U \cup V) \xrightarrow{c_{n+1}} H_n(U \cap V)$$

called connecting maps that makes the following

$$\begin{tikzcd}
 & \cdots \arrow[r, dashed] & H_{n+2}(U \cup V) \arrow[dll, dashed, "c_{n+2}"] \\
H_{n+1}(U \cap V) \arrow[r] & H_{n+1}(U) \oplus H_{n+1}(V) \arrow[r] & H_{n+1}(U \cup V) \arrow[dll, "c_{n+1}"] \\
H_n(U \cap V) \arrow[r, "i_n"] & H_n(U) \oplus H_n(V) \arrow[r, "j_n"] & H_n(U \cup V) \arrow[dll, dashed, "c_n"] \\
H_{n-1}(U \cap V) \arrow[r, dashed] & \cdots &
\end{tikzcd}$$

an exact sequence, where i_n and j_n are the natural maps given by

$$\begin{aligned} i_n : H_n(U \cap V) &\to H_n(U) \oplus H_n(V) \\ a &\mapsto (i_*^{U \cap V \subset U} a, i_*^{U \cap V \subset V} a) \end{aligned}$$

and

$$\begin{aligned} j_n : H_n(U) \oplus H_n(V) &\to H_n(U \cup V) \\ (a, b) &\mapsto i_*^{U \subset U \cup V} a - i_*^{V \subset U \cup V} b. \end{aligned}$$

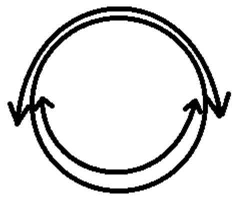
We will discuss the proof of Mayer-Vietoris theorem later when we state it for DeRham Theory.

7.1. **Applications.**

Example 4. Let Y, Z be topological spaces. Consider their direct sum [8] $X = Y \sqcup Z$. Since $Y \cap Z = \emptyset$, by Mayer-Vietoris' Theorem, we have $H_n(X) \simeq H_n(Y) \oplus H_n(Z)$. This also follows from the fact that $C_n(X) = C_n(Y) \oplus C_n(Z)$. $\square$

Example 5. Computing $H_*(S^1)$ Now for a more serious application. Consider the two intervals $U \subset S^1$ and $V \subset S^1$.

[8] The disjoint union of Y and Z, equipped with the topology consisting of open sets of the form $U \cup V$ where $U \subset Y$ and $V \subset Z$ are open.



Both are homeomorphic to an open interval $(0,1) \subset \mathbb{R}$, which is contractible. Hence $H_*(U) \cong H_*(V) \cong H_*((0,1)) \cong H_*(pt)$. And $U \cap V$ is homeomorphic to a disjoint union of two open intervals. So $H_*(U\cap V) \cong H_*(\mathbb{R}) \oplus H_*(\mathbb{R}) \cong H_*(pt) \oplus H_*(pt)$. Since S^1 is connected, we have $H_0(S^1) = \mathbb{R}$. Plugging all this data into the Mayer-Vietoris sequence we get

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & 0\oplus 0 \arrow[r] & H_2(S^1) \arrow[dll, "c_2"] \\
0\oplus 0 \arrow[r] & 0\oplus 0 \arrow[r] & H_1(S^1) \arrow[dll, "c_1"] \\
\mathbb{R}\oplus\mathbb{R} \arrow[r, "i_0"] & \mathbb{R}\oplus\mathbb{R} \arrow[r, "j_0"] & H_0(S^1)=\mathbb{R} \arrow[dll, "c_0"] \\
0 \arrow[r, dashed] & \cdots
\end{tikzcd}$$

Simply by observing the diagram and counting dimensions we get $H_i(S^1) = 0$ for $i > 1$ and $H_1(S^1) \cong \mathbb{R}$. The only non-canonical isomorphism we have here is $H_1(S^1) \cong \mathbb{R}$. To understand this isomorphism better we have to inspect this sequence further.

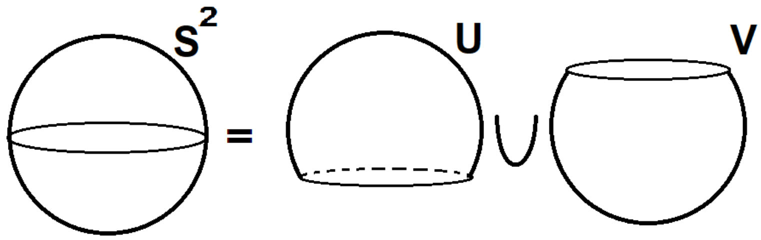
Since $ker\, c_1 = 0$, we may identify $H_1(S^1) = im\, c_1 = ker\, i_0$. Let's introduce some notation to communicate better, denote by $W_1 \sqcup W_2 = U\cap V$ where W_i are the two disjoint intervals. Let $p_i : \Delta^0 \to W_i$ be any specific map. It's clear that each element of $H_0(U \cap V)$ is represented uniquely by a cycle of the form $ap_1 + bp_2$, where $a, b \in \mathbb{R}$. Also, we have $i_*^{U\cap V\subset U} p_1 = i_*^{U\cap V\subset U} p_2 \in H_0(U)$ and $i_*^{U\cap V\subset V} p_1 = i_*^{U\cap V\subset V} p_2 \in H_0(V)$, so $ker\, i_0 = \{ap_1 - ap_2 \in H_0(U \cap V)\}$. There are two natural bases for this space: $p_1 - p_2$ and $p_2 - p_1$. These two choices give us two choices of isomorphisms $H_1(S^1) \cong \mathbb{R}$ and correspond to the two choices of orientation we have on S^1

□

Exercise 22. Compute $H_*(S^n, \mathbb{R})$. □

Example 6. Sketch for $H_*(S^2)$
Consider the following open sets $U, V \subset S^2$.

Since $U, V \cong \mathbb{R}^2$, we have $H_*(U) \cong H_*(V) \cong H_*(\mathbb{R}^2) \cong H_*(pt)$. Notice that the circular belt $U \cap V$ can be retracted onto the equator



of the sphere, which is homeomorphic to S^1. Since this is a homotopy equivalence, by Theorem 2 we have $H_*(U \cap V) \cong H_*(S^1)$. Plugging all we know into the Mayer-Vietoris sequence

$$\begin{array}{ll}
0 \to 0 \oplus 0 \to H_3(S^2) & \cong 0 \\
\to 0 \to 0 \oplus 0 \to H_2(S^2) & \cong \mathbb{R} \\
\to \mathbb{R} \to 0 \oplus 0 \to H_1(S^2) & \cong 0 \\
\to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H_0(S^2) & \cong \mathbb{R}
\end{array}$$

we get that $H_i(S^2) = 0$ for $i < 2$, $H_1(S^2) = 0$ and $H_2(S^2) \cong H_0(S^2) \cong \mathbb{R}$. $\square$

7.2. **Singular Homology Of Manifolds.** From now on, we'll assume that our manifolds are Hausdorff. When we write manifold we will mean a topological manifold below.

Remark 8. We needed this condition for the existence of a triangulation as well. $\square$

Later when we go back to smooth manifolds we'll add another condition, being second countable.

Exercise 23. Find a non-Hausdorff manifold which can be equipped with a smooth atlas. $\square$

Theorem 4. *Let M be an n-dimensional manifold, then $H_i(M; \mathbb{R}) = 0$ for $i > n$.*

Hence the singular homology of M can only live in degrees $i = 0, 1, \ldots, n$.

If we assume that M is also connected, then $H_0(M; \mathbb{R}) \cong \mathbb{R}$ This isomorphism is canonical, where we identify any map from a point to M with $1 \in \mathbb{R}$. It turns out that we know quite a bit about the top degree as well

Theorem 5. *Let M be an n-dimensional manifold. Then*

$$H_n(M;\mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } M \text{ is compact and orientable} \\ 0 & \text{otherwise.} \end{cases}$$

Here the isomorphism with $\mathbb{R}$ depends on our choice of orientation.

Theorem 6 (Poincaré Duality)**.** *Let M be a compact, oriented*[9] *n-dimensional manifold. Then we have the canonical isomorphisms*

$$H_{n-k}(M;\mathbb{R}) \cong \big(H_k(M;\mathbb{R})\big)^\vee$$

for all $k \in \mathbb{Z}$.

Exercise 24. Recall that the linear dual $V^\vee$ of a vector space V is the vector space of linear maps $V \to \mathbb{R}$. Prove the following: $V^\vee \cong V$ if and only if $\dim V < \infty$. The only if part is optional, similar to Exercise 11 □

Exercise 25. Let $f : V \to W$ be a linear map. Define $f^\vee : W^\vee \to V^\vee$ by $f^\vee \alpha = \alpha \circ f$. Show that $f^\vee$ is a linear map. Describe $ker f^\vee$ and $im f^\vee$ in terms of $ker f$ and $im f$.
Let $g : W \to P$ be another linear map, show that $(g \circ f)^\vee = f^\vee \circ g^\vee$. □

7.3. **Singular Cohomology.**

Definition 21. Let X be a topological space. The singular cochain complex $C^*(X;\mathbb{R})$ is defined by

$$C^n(X;\mathbb{R}) = \big(C_n(X;\mathbb{R})\big)^\vee$$

and the coboundary maps $\delta_n = \partial_{n+1}^\vee$ are given by

$$\begin{aligned} \delta_n : C^n(X;\mathbb{R}) &\to C^{n+1}(X;\mathbb{R}) \\ \alpha &\mapsto \alpha \circ \partial_{n+1}. \end{aligned}$$

The cohomology of this complex is called the singular cohomology of X. We denote it by

$$H^*(X;\mathbb{R}) = H^*(C^*(X;\mathbb{R})).$$

□

Exercise 26. Prove that the above defined graded vector space is indeed a cochain complex, *viz* $\delta_{n+1} \circ \delta_n = 0$. □

Exercise 27. Prove that $H^n(X;\mathbb{R}) \cong \big(H_n(X;\mathbb{R})\big)^\vee$. Hint: Start by constructing a map. □

[9]That is, we have chosen a specific orientation and so we are equipped with an isomorphism $H_n(M;\mathbb{R}) \cong \mathbb{R}$.

Exercise 28. Deduce a Mayer-Vietoris sequence for singular cohomology from the Mayer-Vietoris sequence for singular homology. Be careful! □

8. March 21, 2022: Smooth Manifolds, Smooth Maps, Gluing

Recall that for a topological space X, we call $U \subset \mathbb{R}^n$, an open subset $V \subset X$ and a homeomorphism $\varphi : V \to U$ a (coordinate) chart.

Definition 22. With the same notation, V is called the domain of the chart and the functions $x_1, \ldots, x_n$ defined by $x_i : V \xrightarrow{\varphi} U \xrightarrow{pr_i} \mathbb{R}$ are called the cooordinates of the chart. Here pr_i is the projection from $U \subset \mathbb{R}^n$ to the i^{th} Euclidean coordinate. □

Definition 23. Let $\varphi_1 : V_1 \to U_1$ and $\varphi_2 : V_2 \to U_2$ be two charts of the topological space X. The map $\varphi_2 \circ \varphi_1^{-1} = \varphi_1 (V_1 \cap V_2) \to \varphi_2 (V_1 \cap V_2)$ is called the transition map from the chart φ_1 to the chart φ_2.
The charts φ_1 and φ_2 are called smoothly compatible if the transition maps $\varphi_2 \circ \varphi_1^{-1}$ and $\varphi_1 \circ \varphi_2^{-1}$ are both smooth. □

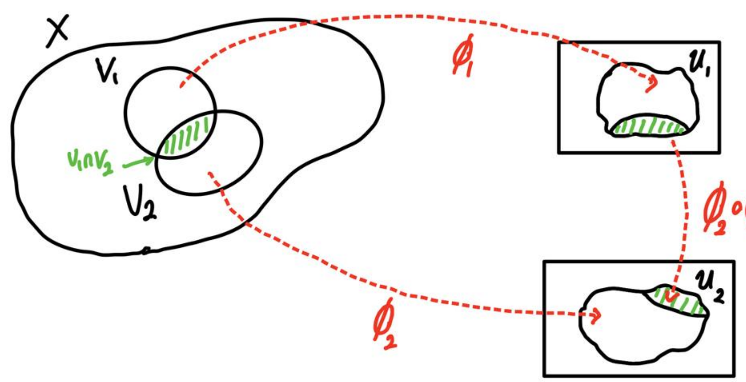


Figure 15. Transition map between charts.

Also recall that a smooth atlas on a topological space X is a collection of charts $\varphi_i : V_i \to U_i|_{i \in I}$ (of the same dimension) such that

- $\cup_{i \in I} V_i = X$
- For any $i, j \in I$, φ_i and φ_j are smoothly compatible.

Warning: There is no such thing as a smooth chart unless we have a smooth atlas.

Definition 24. We call a smooth atlas $\mathcal{A}$ maximal if any chart that is smoothly compatible with $\mathcal{A}$ is already in $\mathcal{A}$. □

Exercise 29. Prove that every smooth atlas is contained in a unique maximal smooth atlas. □

Definition 25. A smooth manifold is a second countable and Hausdorff topological space equipped with a maximal smooth atlas. □

Warning: A maximal smooth atlas is still extra data. We will call it the smooth structure. Let us also call the charts that are in the maximal smooth atlas the smooth charts.

Exercise. Prove that every open subset of $\mathbb{R}^n$ is second countable.

8.1. Examples of Smooth Manifolds.

- $\mathbb{R}^n$
- $S^n \subset \mathbb{R}^{n+1}$
- $\{x^2 + y^3 = 1\} \subset \mathbb{C}^2$
- $Gr(2,4) \coloneqq \{2 \text{ dimensional linear subspaces of } \mathbb{R}^4\}$

What we really mean here is that these have natural topologies which are second countable and Hausdorff and they are equipped with a standard smooth structure.

Exercise 30. Describe each of these topologies and smooth structures. As long as you are correct, you don't need to prove anything. □

- Open subsets of smooth manifolds
- Products of smooth manifolds

Exercise 31. Explain what these mean precisely and actually prove what you wrote. □

8.2. **Smooth Maps.** Recall that for an open subset $U \subset \mathbb{R}^n$, a map $U \to \mathbb{R}$ being smooth means the existence of all iterated partial derivatives; and $U \to V \subset \mathbb{R}^n$ being smooth means that each component is smooth.

Definition 26. Let X be a smooth manifold. We call a function $f : X \to \mathbb{R}$ smooth if for every smooth chart $\varphi : V \to U$, $f \circ \varphi^{-1} : U \to \mathbb{R}$ is smooth. □

Exercise 32. Prove that it suffices to check the smoothness of f on an atlas contained in the maximal smooth atlas. □

Definition 27. Let X, Y be smooth manifolds and $f : X \to Y$ be continuous. We say that f is smooth if for every smooth chart $\varphi_X : V_X \to U_X$ in X and $\varphi_Y : V_Y \to U_Y$ such that $f(V_X) \subset V_Y$, $\varphi_Y \circ f \circ \varphi_X^{-1} : U_X \to U_Y$ is smooth. □

8.3. **Gluing.** Let $\{X_\alpha\}_{\alpha\in I}$ be a collection of topological spaces indexed by a set I. If we are given open subsets $X_{\alpha\beta}\subset X_\alpha$ for every $\alpha\neq\beta\in I$, and homeomorphisms $\varphi_{\alpha\beta}:X_{\alpha\beta}\to X_{\beta\alpha}$ and the following conditions are satisfied, then we call this a gluing data and in particular, $\varphi_{\alpha\beta}$ gluing maps:

- For every $\alpha,\beta\in I,\varphi_{\alpha\beta}\circ\varphi_{\beta\alpha}=id$.
- For every α,β,γ, pairwise distinct,
  $\varphi_{\alpha\beta}\left(X_{\alpha\beta}\cap X_{\alpha\gamma}\right)\subset X_{\beta\alpha}\cap X_{\beta\gamma}$.
- (Cocycle condition)
  $\varphi_{\beta\gamma}\circ\varphi_{\alpha\beta}=\varphi_{\alpha\gamma}$ on $X_{\alpha\beta}\cap X_{\alpha\gamma}$.

Under these assumptions, we can define an equivalence relation on $X=\sqcup_{\alpha\in I}X_\alpha$ by $a\equiv b$ if $a\in X_\alpha, b\in X_\beta$ such that $\alpha\neq\beta$ and $\varphi_{\alpha\beta}(a)=b$. We equip X with its natural topology, that is, the quotient of the disjoint union topology.

Proposition 4. *Assume that each X_α is a smooth manifold, each $\varphi_{\alpha\beta}$ is smooth (as in Definition 27) and X is second countable and Hausdorff. Then, there exists a unique smooth structure on X such that the induced smooth structure on the open subset $X_\alpha\subset X$ is the given one.*

Exercise. Prove this proposition.

Remark 9. I countable $\Rightarrow$ X is second countable. □

Exercise. Prove that every smooth manifold can be obtained by gluing open subsets of $\mathbb{R}^n$.

9. March 24, 2022: Diffeomorphisms, Tangent Bundle of a Manifold, Differential of a Smooth Map

Definition 28. Let X and Y be smooth manifolds, and $f:X\to Y$ be a bijective smooth map. If the inverse map is also smooth, then we call f a diffeomorphism. We also say that X and Y are diffeomorphic. □

Exercise. Let X be Hausdorff and second countable topological space.Let S_1 and S_2 be two maximal smooth atlases. Prove that the identity map $X\to X$ is a diffeomorphism if and only if $S_1=S_2$.

Exercise 33. Consider the real line $\mathbb{R}$ as a topological space. Equip it with (i) its "standard" smooth structure. (ii) smooth structure that admits a chart (U,ϕ) with $U=\mathbb{R}$ and $\phi(x)=x^3$. Prove that (i) and (ii) are not the same smooth structure, but they are diffeomorphic. □

We have talked about the differentiability of maps between smooth manifolds but we didn't take any actual derivatives yet. Note that the partial derivatives of a function as we learned in calculus courses depend on the coordinates that we are given and we do not have such preferred coordinates in a general smooth manifold. We have to develop the notion of tangent bundle to get a head start. Then we will define the differential of a smooth map.

First, we need to deal with the case of open subsets of Euclidean spaces. If you remember your multivariable calculus class well, this is at best a reformulation of what you already know.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets, and $f : U \to V$ be a smooth map. Then the Jacobian matrix of f at a point p is the following matrix.

$$Jac_p(f) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_1}{\partial x_2}(p) & \dots \\ \vdots & \ddots & \\ \frac{\partial f_m}{\partial x_1}(p) & & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}$$

We define the tangent bundle of an open set $U \subset \mathbb{R}^n$ as

$$TU \coloneqq U \times \mathbb{R}^n,$$

which is an open subset of $\mathbb{R}^{2n}$. It is very important to be able to visualize points of TU as a point p in U and a vector at p effectively, see Figure 19.

We define the differential

$$df : TU \to TV$$

of $f : U \to V$ (as above) by formula

$$df(p, v) = (f(p), Jac_p(f)v)$$

Exercise. Prove that if we have open subsets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^k$ and smooth maps $f : U \to V$, $g : V \to W$, we have the following reinterpretation of the chain rule.

$$d(g \circ f) = dg \circ df$$

You can use the multivariable calculus chain rule without proof.

Now we introduce the tangent bundle of an arbitrary smooth manifold.

Remark 10. If $X \subset \mathbb{R}^n$ is a submanifold (will be defined next lecture but you can imagine $S^n \subset \mathbb{R}^{n+1}$), its tangent bundle is the union of tangent spaces at all points of X. See Figure 17. This is where the name "tangent bundle" comes from. The description uses the embedding, which we do not want for a definition. □

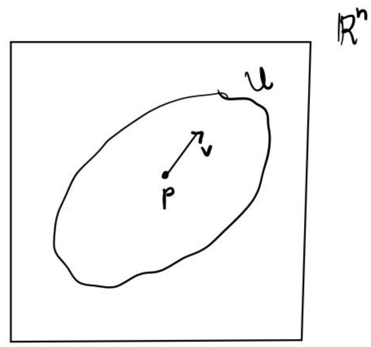


FIGURE 16. A point (p, v) of $TU = U \times \mathbb{R}^n$ can be thought of the vector v at p.

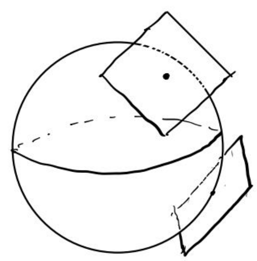


FIGURE 17

Definition 29. Let M be a smooth manifold with maximal atlas $\{\varphi_\alpha : V_\alpha \to U_\alpha\}_{\alpha \in I}$. Define $U_{\alpha\beta} := \varphi_\alpha(V_\alpha \cap V_\beta)$.

We define TM by gluing TU_α along $TU_{\alpha\beta} \subset TU_\alpha$ using

$$d(\varphi_\beta \circ \varphi_\alpha^{-1}) : TU_{\alpha\beta} \to TU_{\beta\alpha}$$

as gluing maps. □

Exercise. Check that this is a valid gluing data.

We also define a smooth map $\pi : TM \to M$ by sending $(p, v) \in TU_\alpha$ to ${\varphi_\alpha}^{-1}(p)$.

Exercise 34. Prove that π is well-defined. Prove that $\forall p \in M$, $\pi^{-1}(p)$ is canonically a $dimM$ dimensional vector space. □

Proposition 5. *Let $f : X \to Y$ be a smooth map between smooth manifolds. Then, there is a canonical smooth map*

$$df : TX \to TY$$

with the following proprieties:

(1) *For any* $\varphi_X : V_X \to U_X$ *and* $\varphi_Y : V_Y \to U_Y$ *coordinate charts such that* $f(V_X) \subset V_Y$, *the diagram*

$$\begin{array}{ccc} TU_X & \longrightarrow & TU_Y \\ \downarrow & & \downarrow \\ TX & \xrightarrow{df} & TY \end{array} .$$

commutes. Here

- *The top map is* $d(\varphi_Y \circ f \circ \varphi_X^{-1})$ *which was already defined.*
- *The vertical maps are the canonical inclusions coming from the gluing construction.*

Clearly, if satisfied, this property determines df *uniquely.*

(2) *The diagram*

$$\begin{array}{ccc} TX & \longrightarrow & TY \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X & \longrightarrow & Y \end{array} .$$

commutes. For all $x \in X$, *the induced map* $\pi_X^{-1}(x) \to \pi_Y^{-1}(f(x))$ *is linear.*

(3) *If* $g : Y \to Z$ *is another smooth map, then* $dg \circ df = d(g \circ f)$. *Finally* $d(id_X) = id_{TX}$.

Exercise 35. Prove part (1) of this theorem. □

Exercise. Prove the remaining parts of this theorem.

Hint: To prove part (1), construct the map df by gluing the maps

$$TU_\alpha \to TU_\beta$$

with $f(V_\alpha) \subset V_\beta$. You need to check that these are compatible with each other. If you can do this one, the other two will be easy.

Remark 11. The map df contains the information of first order derivatives of f. □

Definition 30. X^n smooth manifold

- $TX \xrightarrow{\pi} X$ is called the tangent bundle of X.
- For $x \in X$, the n dimensional real vector space $\pi^{-1}(x)$ is called the tangent space of x and is defined by T_xX.

Let Y be a smooth manifold and $f : X \to Y$ smooth map

- The map $df : TX \to TY$ is called the differential of $f : X \to Y$.
- We obtain linear maps called $df_x : T_xX \to T_{f(x)}Y$ for all $x \in X$.

□

10. March 29, 2022: Submanifolds, Regular Value Theorem

Definition 31. A subset Z of a smooth manifold X^n is called a submanifold of dimension $k \geq 0$ if for every $z \in Z$, there exists a smooth chart $\varphi : V \to U$ of X such that
$$\varphi(V \cap Z) = U \cap (\mathbb{R}^k \times \{0\}) \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n.$$
□

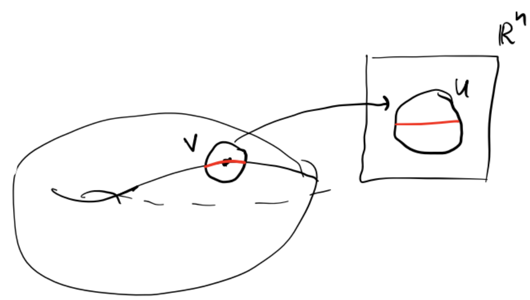


Figure 18

Exercise. Prove that a submanifold $Z \subset X$ with its subspace topology can be equipped with a natural smooth structure such that the inclusion map $Z \hookrightarrow X$ is smooth.

There are two main ways of obtaining submanifolds:

(1) As solutions of smooth equations, i.e. preimages of points in the domain of maps such as
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathbb{R}^2 \\ & Y \end{tikzcd}$$

(2) As subsets parametrized by other manifolds, e.g. $S^2 \to X$.

Let's start with (1). Consider a smooth map $f : X \to Y$. We call $y \in Y$ a *regular value* if for every $x \in X$ such that $f(x) = y$, the linear map $df_x : T_xX \to T_yY$ is surjective.

Theorem 7 (Regular Value Theorem)**.** *Let $f : X \to Y$ be a smooth map. If $y \in Y$ is a regular value, then $f^{-1}(y) \hookrightarrow X$ is a submanifold. Moreover, there are canonical isomorphisms*
$$T_x f^{-1}(y) \cong ker(df_x),$$
for every $x \in f^{-1}(y)$.

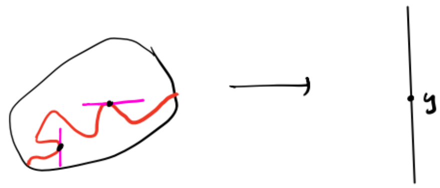


FIGURE 19

Proof Sketch:

- Can easily reduce to $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^k$ open subsets.
- Implicit Function Theorem: We can reorder the coordinates of $\mathbb{R}^n$ such that in an open neighbourhood $x \in U$, $f^{-1}(y) \cap U$ is the graph of a smooth map
$$\varphi = (\varphi_1, \dots, \varphi_k) : W \to \mathbb{R}^k$$
where $W = pr_{n-k}(U) \subset \mathbb{R}^{n-k}$ is open and $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$.
- *Remark* ($\implies$) This already gives a smooth atlas on $f^{-1}(y)$
- Use the map $U \to \mathbb{R}^n$
$(x_1, \dots, x_{n-k}, x_{n-k+1} - \varphi_1(x_1, \dots, x_{n-k}), \dots, x_n - \varphi_k(x_1, \dots, x_{n-k}))$
- This gives a chart by the inverse function theorem.
- For the statement with tangent spaces, note the diagram of smooth maps
$$\begin{tikzcd} f^{-1}(y) \arrow[r] \arrow[d, hook] & * \arrow[d] \arrow[dr] & \\ x \arrow[r] & Y & y \end{tikzcd}$$
and use Exercise 36.

Exercise. Make this statement intuitive for yourself.

Definition 32. A smooth map $f : X \to Y$ is called an immersion if $df_x : T_x X \to T_f(x)Y$ is injective for all $x \in X$. □

Exercise 36. Prove that inclusions of submanifolds into smooth manifolds are injective immersions. □

Example 7. Consider $T^2 = S^1 \times S^1$, which can be represented as the $[0,1] \times [0,1]$ with the ends identified as indicated in Figure 20, and draw on it a line with irrational slope which gives an injective immersion $\mathbb{R} \to T^2$. The image is a dense subset and is not a submanifold. □

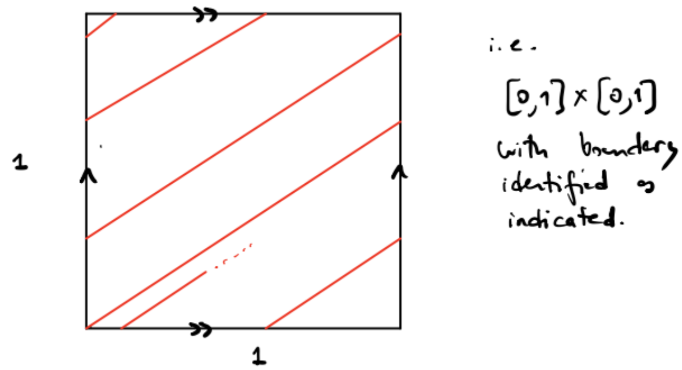


FIGURE 20

Proposition 6. *Assume that $f : Z \to X$ is an injective immersion. Then $f(Z)$ is a submanifold if and only if $Z \to f(Z)$ is a homeomorphism, where $f(Z)$ has the subspace topology.*

Exercise 37. Prove this. □

Proposition 7. *Assume that $f : Z \to X$ is an injective immersion. Then $f(Z)$ is a submanifold and a closed subset if and only if f is a proper map.*

Remark 12. The properness condition in the proposition is automatic if Z is compact. □

Exercise. Find a submanifold of a smooth manifold which is not a closed subset.

Theorem 8 (Whitney Embedding Theorem)**.** *Any smooth manifold X^n can be injectively immersed into $\mathbb{R}^N$ by a smooth proper map for some $N > 0$.*

Remark 13. We can take $N = 2n$, but this is sharp, e.g. $\mathbb{RP}^2$ and Klein bottle. Whitney embedding theorem is not as useful as it might seem because the embeddings are usually inexplicit and complicated. □

11. March 31, 2022: Partitions of Unity

Recall that for a topological space X and a function $f : X \to \mathbb{R}$, support of f is defined as $supp(f) = \overline{\{f(x) \neq 0\}} \subset X$.

Let $B_r(0) \subset \mathbb{R}^n$ be the open ball of radius r centered at the origin.

Lemma 2. *There exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with the following properties.*

(1) $supp(f) \subset B_2(0)$

(2) $f|_{B_1(0)} = 1$
(3) $0 \le f(x) \le 1$, *for all* $x \in \mathbb{R}^n$

It is customary to call such functions bump functions.

Proof. The key is that we can construct a smooth function $g : \mathbb{R} \to \mathbb{R}$ which vanishes on $\mathbb{R}_{\le 0}$ but is positive and increasing on $\mathbb{R}_{>0}$. Here is an example

$$g(x) = \begin{cases} 0 & \text{for } x \le 0 \\ e^{-\frac{1}{x}} & \text{for } x > 0 \end{cases}$$

The smoothness is an easy consequence of the smoothness and the decay of the exponential function e^{-x}.

□

Exercise 38. Using $g(x)$, construct a bump function f. For $n > 1$ you might find it convenient to construct one that only depends on the distance from the origin. □

Remark 14. A special case of the Whitney extension theorem says that for any closed subset $C \subset \mathbb{R}^n$, there exists a smooth function $\mathbb{R}^n \to \mathbb{R}$ which vanishes precisely on C. This becomes useful sometimes. Note that C can be wild, like the Cantor set. □

A collection of subsets of a topological space is called locally finite if for every $x \in X$, there is an open neighbourhood of x intersecting only finitely many members of the collection.

Definition 33. Let X be a smooth manifold and assume that the collection of open subsets $\{U_\alpha\}_{\alpha \in I}$ covers X. We call a collection of smooth functions $\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in I}$ a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in I}$ if the following properties are satisfied,

(1) For every $\alpha \in I$, $supp(f_\alpha) \subset U_\alpha$
(2) $\{supp(f_\alpha)\}_{\alpha \in I}$ is locally finite.
(3) For every $\alpha \in I$ and $x \in X$, $f_\alpha(x) \ge 0$
(4) $\sum_{\alpha \in I} f_\alpha = 1$.

□

Note that the sum in (4) makes sense because of (2). The name comes from (4), where one should think of the constant function 1 as the unit of the algebra of smooth functions on X. If you have a partition of unity subordinate to $\{V_\alpha\}_{\alpha \in I}$, where each V_α is the domain of a chart, we can write any smooth function $q : X \to \mathbb{R}$ as a sum of functions supported inside the domains of those chart, which can then can all thought of as functions defined on $\mathbb{R}^n$,

$$q = 1 \cdot q = \left(\sum_{\alpha \in I} f_\alpha \right) q = \sum_{\alpha \in I} f_\alpha q \,.$$

More often though, you use partitions of unity to patch together locally defined things to a global one. We will see an example soon.

Proposition 8. *Let X be a smooth manifold, and assume that the collection of open subsets $\{U_\alpha\}_{\alpha \in I}$ covers X. Then, there exists a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in I}$.*

Proof. (Sketch) Second countability implies that one can find another cover $\{V_\beta\}_{\beta \in J}$ with the following properties:

(1) J is countable.
(2) For every $\beta \in J$, there exists an $\alpha \in I$ such that $V_\beta \subset U_\alpha$.
(3) For every $\beta \in J$, V_β is the domain of a coordinate chart $\phi_\beta : V_\beta \to \tilde{V}_\beta$, where $\tilde{V}_\beta = B_3(0)$.
(4) For every $\beta \in J$, define $W_\beta = \phi_\beta^{-1}(B_1(0))$. Then, the collection of open sets $\{W_\beta\}_{\beta \in J}$ covers X.
(5) $\{V_\beta\}_{\beta \in J}$, which automatically covers X, is locally finite.

Now let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a bump function as above. Let ρ_β be the extension by zero of $\rho \circ \phi_\beta : V_\beta \to \mathbb{R}$. Define

$$g_\beta := \frac{\rho_\beta}{\sum_{\beta \in J} \rho_\beta} \quad .$$

This is a partition of unity for $\{V_\beta\}_{\beta \in J}$. To find it for $\{U_\alpha\}_{\alpha \in I}$, choose a map $a : J \to I$ such that $V_\beta \subset U_{a(\beta)}$ and set

$$f_\alpha := \sum_{\beta \in J,\, a(\beta) = \alpha} g_\beta \ .$$

□

Definition 34. Let X be a smooth manifold. A Riemannian metric g on X is a smoothly varying positive definite symmetric bilinear form $g_x(\cdot,\cdot)$ on T_xX for every $x \in X$. □

Exercise 39. Define smoothly varying. □

Example 8. $X = U \subset \mathbb{R}^n \implies T_xU = \mathbb{R}^n$ for all $x \in U$. For $v, w \in T_xU$ define $g_x(v,w) = v \cdot w = \sum_{i=1}^n v_i w_i$, which is called the flat metric. □

Proposition 9. *Every smooth manifold admits a Riemannian metric.*

Proof. Let X be our manifold and pick a cover $\{V_\alpha\}_{\alpha\in I}$ by domains of coordinate charts $\varphi_\alpha : V_\alpha \to U_\alpha$ with partition of unity $\{f_\alpha : X \to \mathbb{R}\}_{\alpha\in I}$. Using the flat metric on each U_α we can define a Riemannian metric on V_α called g_α. We define our Riemannian metric by

$$g_x(\cdot,\cdot) = \sum_{\alpha\in I} f_\alpha(x) g_{\alpha,x}(\cdot,\cdot)$$

for all $x \in X$. □

Exercise 40. Check that g is a Riemannian metric. □

Remark 15. A submanifold of $\mathbb{R}^n$ can be equipped with a Riemannian metric by restricting the flat metric to the tangent spaces. It turns out that any Riemannian metric on a smooth manifold X can be obtained by embedding it into an Euclidean space and using this restriction procedure. This is a very difficult theorem called Nash embedding theorem. □

12. April 4, 2022: Vector Bundles

Definition 35. A smooth map (of smooth manifolds) $\pi : E \to B$ is called a vector bundle of rank n if

(1) $\forall b \in B$, $\pi^{-1}(b)$ is equipped with a real vector space structure
(2) $\forall b \in B$, there is an open neighbourhood $b \in U \subset B$ and a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "proj"] \\ & U & \end{tikzcd}$$

such that Φ is a diffeomorphism and $\Phi_b : \pi^{-1}(b) \to \{0\} \times \mathbb{R}^n$ is a linear isomorphism for every $b \in U$.

□

Remark 16. We call such a map Φ a local trivialization and E is called the total space. □

Exercise. Let M^d be a smooth manifold. Prove that $TM \to M$ is a vector bundle of rank d.

Example 9. Mobius bundle: Rank 1 vector bundle over $S^1 = [0,1]/0 \sim 1$ defined as

$$\mathbb{R} \times [0,1]/(s,0) \sim (-s,1) \to S^1,$$

see Figure 21. □

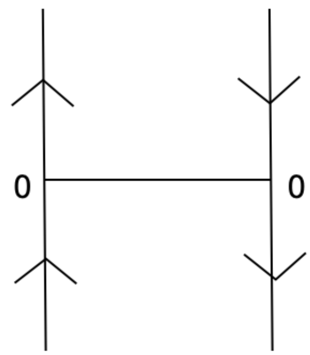


FIGURE 21

12.1. **Constructions of Vector Bundles.**

12.1.1. *Gluing.* Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of a smooth manifold B. Let V be a finite dimensional vector space. Assume that we are given smooth maps

$$t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V) \subset \mathbb{R}^{n^2}$$

for every $\alpha, \beta \in I$ such that ......

Then we can construct a smooth manifold via the formula

$$E := \bigcup_{\alpha\in I} U_\alpha \times V/\sim$$

where $(x, v) \sim (y, w)$ if $x = y$ in B and $t_{\alpha\beta}(x).v = w$.

The canonical map $E \to B$ is a vector bundle of rank $dim(V)$.

Exercise 41. Fill in the blanks and prove the statement. □

The maps $t_{\alpha\beta}$ are called transition functions. Let's reserve the phrase "transition map" to atlases.

Definition 36. A homomorphism/map of vector bundles $E \to B$ and $E' \to B$ over the same B is a map $E \to E'$ which preserves fibers and is fiberwise linear. □

Definition 37. Given vector bundles $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B'$ and smooth map $f : B \to B'$, a smooth map $E \to E'$ is called a vector bundle homomorphism/map covering f if the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

commutes and $\pi^{-1}(b) \to (\pi')^{-1}(f(b))$ is a linear map. $\square$

Exercise. Prove that $df : TX \to TY$ covers $f : X \to Y$.

Exercise. Define what it should mean for two vector bundles over the same base to be isomorphic.

Definition 38. We call a vector bundle $E \to B$ trivial if there is an isomorphism.

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & \mathbb{R}^n \times B \arrow[dl] \\ & B & \end{tikzcd}$$

$\square$

Remark 17. There are many vector bundles that are not trivial, e.g.

- Mobius bundle
- $TS^2 \to S^2$

$\square$

Exercise 42. Prove that $TS^1 \to S^1$ is trivial. $\square$

Definition 39. A section of a vector bundle $E \xrightarrow{\pi} B$ is a smooth map $B \xrightarrow{s} E$ such that $\pi \circ s = id$. $\square$

A section is a choice of a smoothly varying vector at every fiber.

Trivial vector bundles have many sections which are not zero anywhere. The non-trivial bundle examples we gave do not have any such sections.

Sections of $TM \to M$ are called vector fields of M.

Theorem 9 (Hairy Ball Theorem)**.** S^2 *does not have a non-vanishing vector fields.*

To understand the name properly let us make something from the previous class more explicit.

Definition 40. Let $E \to B$ vector bundle, $E_b := \pi^{-1}(b)$. We call $S \subset E$ a subbundle, if

(1) For every $b \in B$, $S \cap E_b \subset E_b$ is a subspace
(2) $S \subset E$ is a submanifold
(3) $\pi|_S : S \to B$ is a vector bundle.

$\square$

Remark 18. I remember proving that (1) and (2) implies (3). $\square$

Exercise 43. Let $Z \subset X$ be submanifold, $\imath : Z \hookrightarrow X$ be inclusion map. Define $\imath^*TX := \pi^{-1}(Z)$, where $\pi : TX \to X$. Prove

(1) $\imath^*TX \to Z$ is a vector bundle
(2) $S := \bigcup_{z \in Z} im(di_z) \subset \imath^*TX$ is a subbundle.
(3) $S \to Z$ and $TZ \to Z$ are canonically isomorphic vector bundles.

$\square$

This means that for example we can think TS^2 as the union of tangent planes of $S^2 \subset \mathbb{R}^3$; a vector field on S^2 as a collection of smoothly varying tangent vectors at every point of S^2.

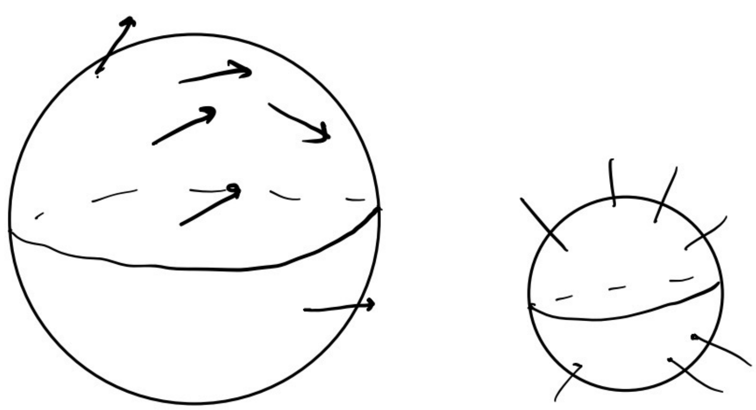


FIGURE 22. arrows = hair after being combed
hairy ball theorem = you can not comb the hair without creating discontinuity

Remark 19. It is customary among non-geometers to work only with the sections of a vector bundle and never talk about the vector bundle itself. Here you think of sections as a collection of local vector valued functions, which transform according to some rules (i.e. the transition functions, which in case of vector bundles related to the tangent bundle can be expressed in terms of changes of coordinates - this expression transforms as xxx, Einstein conventions etc.) I think it is a shame and the only reason to do this could be that the mental effort to conceptualize a non-trivial bundle is non-trivial. This is similar to the insistence of some physicists to never talk about the flow of a vector field but only individual solutions of the corresponding ODE. Neither of these geometric notions (flows and global bundles) will help if all you want is to compute something, but they definitely help in thinking about what you are doing when you are doing the computation. Laziness turns into a defense mechanism that causes people to think mathematicians are just being fancy. □

There is also a converse to gluing, namely given a vector bundle $\pi : E \to B$, you choose an open cover $\{U_\alpha\}_{\alpha \in I}$ of B with trivializations

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$$

We obtain $\tilde{t}_{\alpha\beta} := \Phi_\beta \circ \Phi_\alpha^{-1}$

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^n \xrightarrow{\sim} (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

or equivalently $t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^n)$. We can use this data to glue a new vector bundle that is canonically isomorphic to $E \to B$.

13. April 7, 2022: More on Vector Bundles, Cotangent Bundle, Orientations

I want to start by noting a useful result about subbundles.

Proposition 10. *Assume that we have a vector bundle homomorphism*

$$\begin{tikzcd} \phi : E \arrow[rr] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd}$$

If $\phi_b : E_b \to E'_b$ has the same rank for all $b \in B$, then

(1) $\bigcup_{b \in B}$ *ker* $\phi_b \subset E$
(2) $\bigcup_{b \in B}$ *im* $\phi_b \subset E'$

are both subbundles.

Example 10. The constant rank assumption is not always satisfied. For example consider the map of vector bundles:

$$\begin{tikzcd}
(x,t) \arrow[rr, mapsto, phantom, "\longmapsto"] & & (x, x.t) \\
\mathbb{R}_t \times \mathbb{R}_x \arrow[rr] \arrow[dr] & & \mathbb{R}_t \times \mathbb{R}_x \arrow[dl] \\
& \mathbb{R}_x &
\end{tikzcd}$$

One can immediately check that the conclusion of the proposition does not hold. □

Let's now assume that we have the vector bundle $E \to B$ constructed by gluing $\{U_\alpha \times V \to U_\alpha\}_{\alpha \in I}$ where V is a finite dimensional vector space and $\{U_\alpha\}_{\alpha \in I}$ is an open cover of B with the transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ for every $\alpha, \beta \in I$.

Now let W be another finite dimensional vector space and

$$\Psi : \mathrm{GL}(V) \to \mathrm{GL}(W)$$

be a group homomorphism.

Using Ψ we can construct a new vector bundle by gluing $\{U_\alpha \times W \to U_\alpha\}_{\alpha \in I}$ with the transition maps

$$\psi \circ t_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(W)$$

.

Exercise. Prove that this is a valid gluing data.

13.1. **The Dual Vector Bundle.** Let us consider the case

$$W = V^\vee := \mathrm{Hom}(V, \mathbb{R})$$

and

$$\begin{aligned} \mathrm{GL}(V) \to \mathrm{GL}(V^\vee) &\to \mathrm{GL}(V^\vee) \\ A \mapsto A^* & \\ & B \mapsto B^{-1} \end{aligned}$$

where the first map takes the dual map of a linear map.

Remark 20. If we choose a basis for V, then we get $V \cong \mathbb{R}^n$, $V^\vee \cong \mathbb{R}^n$ and $\mathrm{GL}(\mathbb{R}^n) \cong$ invertible nxnmatrices . Then the first map takes the transpose and the second inverts. □

These are both anti-homomorphisms and the composition is a homomorphism.

$\implies$ We get a new vector bundle $\pi^\vee : E^\vee \to B$.

Proposition 11. *There are canonical isomorphisms* $(E^\vee)_b \cong (E_b)^\vee$.

Exercise 44. Prove this. $\square$

Definition 41. The dual vector bundle to $TM \to M$ is called the cotangent bundle and is denoted by $T^*M \to M$. We will denote $(T^*M)_b$ by T_b^*M and identify it with $T_bM^\vee$ using the canonical map.

Elements of T_b^*M are called covectors at b and sections of $T^*M \to M$ are called covector fields or differential 1-forms. $\square$

13.2. **The Determinant Vector Bundle.** Now consider the case $V = \mathbb{R}^n$, $W = \mathbb{R}$ and

$$\begin{aligned} \mathrm{GL}(\mathbb{R}^n) &\to \mathrm{GL}(\mathbb{R}) \\ A &\mapsto \det(A) \cdot . \end{aligned}$$

For example, we can apply this to $TM \to M$ and obtain the line bundle $det(TM) \to M$.

Remark 21. In the next lecture, we will define the exterior powers of a vector space, and in particular

$$\det(V) \coloneqq \wedge^{\dim(V)} V.$$

It will be the case that

$$\det(TM)_b \underset{\text{can.}}{\cong} \det(T_bM).$$

$\square$

13.3. **Orientation.** Any two bases e, f of a finite dimensional vector space V are related by a unique change of basis matrix $A(e, f)$ whose entries are defined by $e_i = \Sigma_j A_{ji} f_j$. We say that e and f are positively related if $\det A(e, f) > 0$.

This divides the set of bases of V into two groups so that members of each group are pairwise positively related. Let $o(V)$ be the set whose elements are the two groups.

Definition 42. An orientation of V is a choice of an element of $o(V)$ as the positively oriented bases and the other as the negatively oriented bases. $\square$

Note that if V is one-dimensional,

$$o(V) \underset{can.}{\cong} V - \{0\}/\sim,$$

$v \sim v'$ if $v = cv'$ for $c > 0$.

Let $\pi : E \to B$ be a line bundle and let $Z_B \subset E$ be the image of the zero section. We can construct a $2:1$ covering space $o(\pi) : o(E) \to B$ by

$$o(E) := {E \setminus Z_B}/{\sim},$$

with $v \sim v'$ if $v, v' \in E_b$ for some $b \in B$ and $v = cv'$ with $c > 0$.

Exercise 45. Prove that the canonical map $o(E) \to B$ is a $2:1$ covering space. □

By construction $o(E)$ is canonically identified with $\bigcup_{b\in B} o(E_b)$. A section of $o(E) \to B$ is nothing but a continuously varying choice of orientations for each E_b.

Of course such a section does not always exist – equivalent to $2:1$ cover being trivial.

Example 11. The orientation bundle of the Mobius bundle is isomorphic to the nontrivial cover $S^1 \to S^1$, $\theta \mapsto 2\theta$ □

If there is a continuous section of its orientation bundle we call $E \to B$ orientable and the choice of a section an orientation of $E \to B$.

Definition 43. M is called orientable if $\det(TM) \to M$ is orientable. An orientation of this bundle is called an orientation of M. □

Remark 22. Next Lecture: For any finite dimensional vector space V,

$$o(V) \underset{can.}{\cong} o(\det(V)).$$

□

$\implies$ An orientation of M is equivalent to a continuously varying choice of orientations for each tangent space T_xM, $x \in M$.

Definition 44. If $U \subset \mathbb{R}^n$ open, the standard orientation of U is the one where the basis $e_1, \ldots, e_n$ of T_bU is positive for all $b \in U$. □

Proposition 12. *Let M be a smooth manifold. M is orientable if and only if there exists a subatlas $\{\phi_\alpha : V_\alpha \to U_\alpha\}_{\alpha\in I}$ of the smooth structure such that the transition map $\phi_\beta \circ \phi_\alpha^{-1} : U_{\alpha\beta} \to U_{\beta\alpha}$ satisfies*

$$det(Jac_p(\phi_\beta \circ \phi_\alpha^{-1})) > 0$$

for all $p \in U_{\alpha\beta}$, where $U_{\alpha\beta} := \phi_\alpha(V_\alpha \cap V_\beta)$.

Definition 45. Let us call such a subatlas an oriented subatlas. □

14. April 11, 2022: Constructing Vector Spaces, The Exterior Algebra

Definition 46. Let V and W be vector spaces. The direct sum $V \oplus W$ is the vector space with underlying set $V \times W$ endowed with the vector space structure given by

$$\begin{aligned} c(v, w) &= (cv, cw) \\ (v, w) + (v', w') &= (v + v', w + w') \end{aligned}$$

$\square$

If $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ are bases of V and W, respectively, then $\tilde{e}_i \coloneqq (e_i, 0)$ and $\tilde{f}_j \coloneqq (0, f_j)$ together form a basis of $V \oplus W$.

Exercise 46. Let $E \to B$ and $E' \to B$ be vector bundles. Define the Whitney sum vector bundle

$$E \oplus E' \to B$$

such that each fiber $\left(E \oplus E'\right)_b$ is canonically identified with $E_b \oplus E'_b$. $\square$

14.1. Tensor Products.

We'll now define the tensor product $V \otimes W$ of two vector spaces V and W.

Let $T(V, W)$ be the vector space generated by the set $V \times W$. Let $Q(V, W) \subset T(V, W)$ be the smallest subspace that contains

$$\begin{aligned} c(v, w) - (cv, w) \\ c(v, w) - (v, cw) \\ (v + v', w) - (v, w) - (v', w) \\ (v, w + w') - (v, w) - (v, w') \end{aligned}$$

for all $c \in \mathbb{R}$ and $v, v' \in V$ and $w, w' \in W$. Define

$$V \otimes W \coloneqq \frac{T(V, W)}{Q(V, W)}.$$

The equivalence classes $[(v, w)]$ are denoted by $v \otimes w$ and are called pure tensors. Every element of $V \otimes W$ is a finite sum of pure tensors.

Exercise 47. Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be bases of V and W, respectively. Show that $e_i \otimes f_j : 1 \leq i \leq n, 1 \leq j \leq m$ is a basis of $V \otimes W$. In particular $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm}$. $\square$

Tensor products can be tricky to work with but it will be worth our time.

Lemma 3. *Given linear maps $f : V \to W$ and $f' : V' \to W'$, there exists a unique map*

$$f \otimes f' : V \otimes V' \to W \otimes W'$$

which satisfies

$$f \otimes f' : v \otimes v' \mapsto f(v) \otimes f'(v').$$

Proof. Define a linear map

$$\begin{aligned} \phi : T(V, V') &\to T(W, W') \\ (v, v') &\mapsto (f(v), f'(v')) \end{aligned}$$

by linearly extending. All we have left to check is that $\phi(Q(V, V')) \subset Q(W, W')$, which is easy. $\square$

Exercise 48. Let $A : \mathbb{R}^n \to \mathbb{R}^{n'}$ and $B : \mathbb{R}^m \to \mathbb{R}^{m'}$ be linear maps. Describe the matrix of $A \otimes B$ in terms of the matrices of A and B. $\square$

Exercise. Suppose that in the above exercise $n = n'$ and $m = m'$, that is, A and B are endomorphisms. Then $A \otimes B$ is also an endomorphism. Describe

- The trace $\mathrm{tr} A \otimes B$ in terms of $\mathrm{tr} A$ and $\mathrm{tr} B$.
- The determinant $\det A \otimes B$ in terms of $\det A$ and $\det B$.
- The eigenvalues/vectors of $A \otimes B$ in terms of those of A and B.

Exercise. Let $E \to B$ and $E' \to B$ be vector bundles. Define the vector bundle $E \otimes E' \to B$ so that $\big(E \otimes E'\big)_b$ is canonically isomorphic to $E_b \otimes E'_b$.

Lemma 4. *There is a unique isomorphism*

$$\begin{aligned} \big(U \otimes V\big) \otimes W &\to U \otimes \big(V \otimes W\big) \\ (u \otimes v) \otimes w &\mapsto u \otimes (v \otimes w). \end{aligned}$$

Proof. Routine but tedious to write down. Define the only possible map and see that it works. $\square$

Definition 47. With this in mind we can define inductively

$$V^{\otimes(k+1)} := V \otimes V^{\otimes k}.$$

$$v_0 \otimes v_1 \otimes \cdots \otimes v_k := v_0 \otimes (v_1 \otimes \cdots \otimes v_k).$$

by taking $V^{\otimes 0} = \mathbb{R}$. Also, we take

$$U \otimes V \otimes W := U \otimes \big(V \otimes W\big)$$

$$u \otimes v \otimes w := u \otimes (v \otimes w)$$

$\square$

14.2. **The Exterior Algrebra.** We have finally arrived at the exterior product operation.

Definition 48. Let $k \geq 1$, we define

$$\Lambda^k V = {V^{\otimes k}}\big/{S_k}$$

where S_k is the smallest subspace of $V^{\otimes k}$ that contains the set

$$\{v_1 \otimes \cdots \otimes v_k \in V^{\otimes k} : v_1, \ldots, v_k \text{ are linearly dependent}\}.$$

We denote the class of $v_1 \otimes \cdots \otimes v_k$ by $v_1 \wedge \cdots \wedge v_k$. The space $\Lambda^k V$ is called the kth exterior product of V. We also take $\Lambda^0 V = V^{\otimes 0} = \mathbb{R}$. Also notice that $\Lambda^k V = 0$ for $k > \dim V$. □

Remark 23. The space S_k could also be described to be the smallest subspace of $V^{\otimes k}$ that contains all elements of the form $\cdots \otimes v \otimes v \otimes \cdots \in V^{\otimes k}$. One could also see that $\bigoplus_{k=1}^{\infty} S_k$ is a two-sided ideal in the graded algebra(to be defined later)

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

that is generated by $v \otimes v : v \in V$. □

Proposition 13. *Let $e_1, \ldots, e_n$ be a basis of V. Let I_k be the set of order preserving functions*

$$i : \{1, \ldots, k\} \to \{1, \ldots, n\}.$$

Then

$$\{e_{i(1)} \wedge \cdots \wedge e_{i(k)}\}_{i \in I_k}$$

is a basis of $\Lambda^k V$.

We'll omit the proof for now. The statement is much more important.

Definition 49. A graded algebra is a graded vector space $\{V_n\}_{n \in \mathbb{Z}}$ and a collection of bilinear maps

$$\phi_{n,m} : V_n \times V_m \to V_{n+m}.$$

A graded algebra gives an algebra structure on

$$V := \bigoplus_{n \in \mathbb{Z}} V_n$$

by defining

$$\Big(\sum_{n \in \mathbb{Z}} v_n\Big) \cdot \Big(\sum_{m \in \mathbb{Z}} w_m\Big) = \sum_{n,m \in \mathbb{Z}} \phi_{n,m}(v_n, w_m).$$

Note that the right hand side has V_k component given by

$$\sum_{n+m=k} \phi_{n,m}(v_n, w_m).$$

This is non-zero only for finitely many $k \in \mathbb{Z}$.

Most of the terminology of algebras carry over, *e.g* associative and unital algebras **BUT** commutative means that for $x \in V_n$, $y \in V_m$ we have that

$$x \cdot y = (-1)^{nm} y \cdot x.$$

$\square$

Proposition 14. *The graded vector space $\{\Lambda^k V\}_{k\geq 0}\}$ forms a commutative graded algebra together with the product*

$$\Lambda^k V \times \Lambda^l V \to \Lambda^{k+l} V$$

$$(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_l) \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l.$$

This is called the exterior algebra or the Grassmann algebra of V.We denote

$$\Lambda^* V = \bigoplus_{k=0}^{\infty} \Lambda^k V.$$

The product we defined here is called the exterior product, which we'll denote by $\wedge$.

Exercise 49. Write a concrete description of $\Lambda^* \mathbb{R}^n$ for $n = 0, 1, 2, 3, 4$.
$\square$

15. April 14, 2022: More on the Exterior Algebra, Wedge Product, Alternating Multilinear Maps

Recall: For a vector space V,

- $\Lambda^k V$ is the k^{th} exterior power,
- $\Lambda^k V = 0$ for $k > dimV$,
- $\Lambda^0 V \cong \mathbb{R}, \Lambda^1 V \cong V$,
- Assume V is finite dimensional and let $e_1, \ldots, e_n$ be a basis. Let

$$I_k \coloneqq \{\{1, \ldots, k\} \to \{1, \ldots, n\} \text{ injective, order-preserving maps.}\}.$$

Then, $\{e_{i(1)} \wedge \ldots \wedge e_{i(k)}\}_{i\in I}$ forms a basis of $\Lambda^k V$. In particular $\Lambda^{dimV} V \cong \mathbb{R}$, but this isomorphism is not canonical since we have chosen a basis.

Definition 50. Let V be a finite dimensional vector space. We define $\det V \coloneqq \Lambda^{dimV} V$. Choosing a connected component of $\Lambda^{dimV} V/\{0\}$ is the same thing as choosing an element of $o(V)$. $\square$

Proposition 15. *Given a linear map $f : V \to W$, there is a unique homomorphism*

$$\Lambda^k V \to \Lambda^k W \text{ such that}$$
$$v_1 \wedge \ldots \wedge v_k \mapsto f(v_1) \wedge \ldots \wedge f(v_k).$$

Proof. We already have a canonical map $V^{\otimes k} \to W^{\otimes k}$ such that $v_1 \otimes \ldots \otimes v_k \mapsto f(v_1) \otimes \ldots \otimes f(v_k)$. We need to show that if $v_1, \ldots, v_k$ is linearly dependent, then so is $f(v_1), \ldots, f(v_k)$, and this is true by linearity. □

Exercise 50. If $f : V \to V$ is a linear map, we can define $\det(f) \in \mathbb{R}$. Prove that the induced map $\Lambda^{dimV} V \to \Lambda^{dimV} V$ is given by multiplication with $\det(f)$. □

15.1. **Wedge product.** We define the wedge product, also denoted by $\wedge$ as

$$\wedge : \Lambda^k V \times \Lambda^{k'} V \to \Lambda^{k+k'} \text{ such that}$$
$$(v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_{k'}) \mapsto v_1 \wedge \ldots \wedge v_k \wedge w_1 \wedge \ldots \wedge w_{k'}.$$

$\wedge$ is graded commutative: for $\alpha \in \Lambda^k V, \beta \in \Lambda^k V$, we have

$$\alpha \wedge \beta = (-1)^{kk'} (\beta \wedge \alpha).$$

The algebra

$$\Lambda^* V = \oplus_{k=0}^{\infty} \Lambda^k V$$

is called the exterior algebra. Recall Definition 49.

Remark 24. We have another graded algebra $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$, defined by

$$\otimes : V^{\otimes k} \times V^{\otimes k'} \to V^{\otimes k+k'},$$

where

$$(v_1 \otimes \ldots \otimes v_k, w_1 \otimes \ldots \otimes w_k) \mapsto v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_k.$$

For $v \neq w$ nonzero elements of V, $v \otimes w \neq \pm w \otimes v$ so this algebra is not commutative. It is also infinite dimensional unless $V = \{0\}$. Quotient map $T(V) \to \Lambda^* V$ is an algebra homomorphism. □

Proposition 16. *Given a linear map $f : V \to W$, there is a unique homomorphism for every $k \geq 0$,*

$$\Lambda^k V \to \Lambda^k W \text{ such that}$$
$$v_1 \wedge \ldots \wedge v_k \mapsto f(v_1) \wedge \ldots \wedge f(v_k).$$

This map respects the wedge product.

Proof. We alredy have a map $V^{\otimes k} \to W^{\otimes k}$ satisfying $v_1 \otimes \ldots \otimes v_k \mapsto f(v_1) \otimes \ldots \otimes f(v_k)$. We need to show that if $v_1, \ldots, v_k$ is linearly dependent, then so is $f(v_1), \ldots, f(v_k)$. This is true by linearity. $\square$

15.2. **Alternating Multilinear Maps.** Let W be a finite dimensional vector space and $V := W^V = \mathrm{Hom}(W, \mathbb{R})$. We will give a more concrete description of $\Lambda^* V$.

Definition 51. A multilinear map $\alpha : W \times \ldots \times W \to \mathbb{R}$ is called alternating if $\alpha(\ldots, w_i, w_{i+1} \ldots) = -\alpha(\ldots, w_{i+1}, w_i \ldots)$. $\square$

Proposition 17. *Let $V = W^V$ be as above. Then, $\Lambda^k V$ is canonically isomorphic to the vector space $Alt^k(W)$ of alternating multilinear maps $W \times \ldots \times W \to \mathbb{R}$. The isomorphism sends $v_1 \wedge \ldots \wedge v_k$ to the element of $Alt^k(W)$ defined by*

$$(w_1, \ldots, w_k) \mapsto \sum_{\sigma \in \Sigma_k} sign(\sigma) v_1(w_{\sigma(1)}) \ldots v_k(w_{\sigma(k)})$$

where Σ_n is the set of all permutations $\{1, \ldots, k\} \to \{1, \ldots, k\}$.

Note that the multilinearity and alternating properties are obvious from the definition. It is also not too difficult to check that the map is well-defined.

Exercise 51. Prove that this map is an isomorphism.
Hint: One option is to show surjectivity (or injectivity) and do a dimensioun count. $\square$

The isomorphism induces a map

$$\wedge : \mathrm{Alt}^k W \times \mathrm{Alt}^{k'} W \to \mathrm{Alt}^{k+k'} W$$
$$(\alpha, \beta) \mapsto \alpha \wedge \beta,$$

where direct (but confusing) computation shows

$$\alpha \wedge \beta(w_1, \ldots, w_{k+k'}) = \sum_{\sigma \in Sh(k,k')} sign(\sigma) \alpha(w_{\sigma(1)}, \ldots, w_{\sigma(k)}) \beta(w_{\sigma(k+1)}, \ldots, w_{\sigma(k+k')})$$

where $Sh(k, k') \subset \Sigma_{k+k'}$ that satisfy $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(k+k')$. This product is sometimes called the shuffle product.

Remark 25. $Sh(k, l)$ is the set of all ways of taking $k + l$ cards, and shuffling the first k cards and the last l cards. $\square$

Instead of a proof, we give an example. Assume that V is a four dimensional vector space with basis $e_1, \ldots, e_4$. Let's check that the diagram commutes when we input $e_1^\vee \wedge e_2^\vee$ and $e_3^\vee \wedge e_4^\vee$ in the diagram

$$\begin{tikzcd}
\Lambda^2 V \times \Lambda^2 V \arrow[rr, "\wedge"] \arrow[d, "\wr\|"'] & & \Lambda^4 V \arrow[d, "\wr\|"'] \\
\mathrm{Alt}^2(W) \times \mathrm{Alt}^2(W) \arrow[rr, "\text{Shuffle}"] & & \mathrm{Alt}^4(W)
\end{tikzcd}$$

We just need to follow the definitions here. Note that in order to specify an alternating multilinear map with k inputs, we only need to say what it does on all the inputs of the form $e_{i(1)}, \dots, e_{i(k)}$ with $i : [k] \to [4]$ increasing and injective. Below we only write the inputs where the result is non-zero.

$$e_1^\vee \wedge e_2^\vee \mapsto ((e_1, e_2) \mapsto 1)$$

$$e_3^\vee \wedge e_4^\vee \mapsto ((e_3, e_4) \mapsto 1)$$

$$e_1^\vee \wedge e_2^\vee \wedge e_3^\vee \wedge e_4^\vee \mapsto ((e_1, e_2, e_3, e_4) \mapsto 1)\,.$$

$$\text{Shuffle}\left(((e_1, e_2) \mapsto 1)\,, ((e_3, e_4) \mapsto 1)\right) = ((e_1, e_2, e_3, e_4) \mapsto 1)\,.$$

I leave it to you to fill in the words.

Remark 26. We will freely use the isomorphisms $\Lambda^k(W^V) \cong \mathrm{Alt}^k(W)$, when W is finite dimensional. $\square$

16. April 18, 2022: Differential Forms, Pull-back

15 Minutes for Questions. Let W be a finite dimensional vector space and $V := W^\vee$

$$\begin{tikzcd}
W \arrow[r] \arrow[dr] & W^\vee \arrow[r] & \Lambda^k(V) \arrow[dl, equal] \\
& Alt^k(W) &
\end{tikzcd}$$

Here by the equal sign we mean that there is a an isomorphism between the two that we agreed to use.

Reminding you again that $\Lambda^0 V := \mathbb{R}$,

$$\wedge : \Lambda^0 V \times \Lambda^k V \to \Lambda^k V \quad ?$$

Think of $Alt^0(W)$ as a multi-linear map with 0-inputs, which is simply a real number.

Exercise 52. Let $e_1, e_2, ..., e_n$ be basis of W and let $e_1^\vee, e_2^\vee, ..., e_n^\vee$ be dual basis of $V = W^\vee$. Compute

$$e_1 \wedge ... \wedge e_n(w_1, ..., w_n)$$

for arbitrary elements $w_1, w_2, .., w_n \in W$ by writing them as a linear combination of basis elements. Assuming that $W = \mathbb{R}^n$ and $e_1, .., e_n$ is standard basis interpret your result as a geometric quantity related to $w_1, w_2, .., w_n$ as vectors in the Euclidian space.
(After some computation it can be seen that $e_1 \wedge ... \wedge e_n(w_1, ..., w_n)$ is equal to the determinant of the matrix consisting of the coefficients of w_i's in standard basis. Therefore the resulting quantity will be the volume of n-parallelepiped generated by w_i's.) □

If you have a linear map $f : W \to U$, you can define

$$f^* : \Lambda^*(U^\vee) \to \Lambda^*(W^\vee)$$

by the formula

$$f^*\alpha(w_1, .., w_n) = \alpha(f(w_1), .., f(w_n)).$$

This is the same operation as the one induced from $f^* : U^\vee \to W^\vee$ on $\Lambda^k U^\vee \to \Lambda^k W^\vee$.

Given a vector bundle $E \to B$ we can construct a vector bundle $\Lambda^k E \to B$ whose fibers are canonically identified with $\Lambda^k E_B$.

How do we check that

$$GL(V) \to GL(\Lambda^k V)$$

is a group homomorphism?

16.1. **Differential k-forms.**

Definition 52. Let M be a smooth manifold. A differential k-form is a smooth section of $\Lambda^k T^*M \to M$ □

Note that we can think of a differential k-form as a collection of smoothly varying alternating multi-linear maps $\underbrace{T_xM \times ... \times T_xM}_{k} \to \mathbb{R}$ for $\forall x \in M$.

We denote the vector space of all differential k-forms by $\Omega^k(M)$. We have a graded commutative algebra $\Omega^*(M) := \bigoplus_{k=0}^{\infty} \Omega^k(M)$. What is $\Omega^0(M)$?

16.2. **Pull-back of Differential Forms.**

Definition 53. Given a smooth map $f : X \to Y$ and a differential k-form α on Y we can define a differential k-form $f^*\alpha$ on X by

$$(f^*\alpha)_p(v_1, .., v_k) = \alpha_{f(p)}(df_p v_1, ..., df_p v_n).$$

This is called the pullback of α by f.

$$f : X \to Y \rightsquigarrow f^* : \Omega^*(Y) \to \Omega^*(X)$$

$\square$

Proposition 18. *If $g : Y \to Z$ is also smooth*

$$(g \circ f)^* = f^* \circ g^*$$

Exercise 53. Prove the proposition above. $\square$

Proposition 19. *f^* preserves the wedge product of differential forms.*

Exercise 54. Prove the proposition above. $\square$

Exercise 55. Describe the pull-back operation when $k = 0$. $\square$

17. April 21, 2022: Differential Forms on Open Subsets of the Euclidean Space, Computing Pullback in Coordinates, Directional Derivative

17.1. **Differential Forms on Open Subsets of $\mathbb{R}^n$.**

Let $U \subset \mathbb{R}^n$ be open. Let's start with vector fields, $TU \simeq U \times \mathbb{R}^n$, $x_1, \cdots, x_n$ are coordinates on U. Then, $\frac{\partial}{\partial x_i}$ is defined to be the constant vector field equal to $e_i = (0, \cdots, 1, \cdots, 0)$ at every $p \in U$. Then $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ gives a basis of T_pU at every $p \in U$.

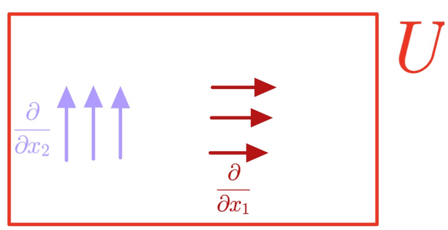


Figure 23. Constant Vector Field

Every vector field on U can be written uniquely as $\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ where $f_i : U \to \mathbb{R}$ are smooth.

Now, we move on to covector fields (alternatively called differential 1-forms/sections of $T^*U \to U$/elements of $\Omega^1(U)$): We have $T^*U \simeq U \times (\mathbb{R}^n)^\vee$. Let dx_i be the constant covector field equal to $e_i^\vee$ at every $p \in U$ and $dx_1, ..., dx_n$ gives a basis of T_p^*U at every $p \in U$.

Every covector field on U can be written uniquely as $\sum\limits_{i=1}^{n} f_i dx_i$ where $f_i : U \to \mathbb{R}$ are smooth.

Note. $e_i^\vee \wedge e_j^\vee = -e_j^\vee \wedge e_i^\vee \in \Lambda^2(\mathbb{R}^n)^\vee$. This implies $dx_i \wedge dx_j = -dx_j \wedge dx_i \in \Omega^2(U)$.

Finally getting to general differential forms, note

$$\Lambda^k T^*U \cong U \times \Lambda^k((\mathbb{R}^n)^\vee)$$

We define the differential k-form $dx_I \coloneqq dx_{I(1)} \wedge ... \wedge dx_{I(k)}$ for $I : \{1, ..., k\} \to \{1, ..., n\}$. Every differential k-form on U can be written uniquely as $\sum\limits_{I \in I_k} f_I dx_I$, $f_I : U \to \mathbb{R}$ smooth, and $I_k = \{I : \{1, ..., k\} \to \{1, ..., n\} : I \text{ strictly increasing}\}$.

Example 12. $U \subset \mathbb{R}^3$; coordinates are x, y, z. Every differential two form on U can be written as $f dx \wedge dy + g dy \wedge dz + h dx \wedge dz$ (uniquely). $\square$

Exercise. Write $(\sum_{i=1}^3 f_i dx_i) \wedge (\sum_{i=1}^3 g_i dx_i)$ in this form.

17.2. **Computing Pullback in Coordinates.** Let $U \underset{x_1,...x_n}{\subset} \mathbb{R}^n$, $V \underset{y_1,...y_m}{\subset} \mathbb{R}^m$ be open, and $\varphi : V \to U$ be a smooth map. Let $\alpha = \sum\limits_{I \in I_k} f_I dx_I$ be a differential k-form on U. Let us write down $\varphi^*\alpha$ in coordinates $y_1, ...y_m$. The case for 0-forms is as follows: $\alpha = f : U \to \mathbb{R} \implies \varphi^* f = f \circ \varphi$. How about $\varphi^* dx_i$? It should be of the form $\sum_{j=1}^m \square dy_j$, where $\square$ is to be calculated as below:

$$\varphi^* dx_i(\frac{\partial}{\partial y_j})(p) = dx_i(d\varphi_p \frac{\partial}{\partial y_j}) = dx_i(\sum_{k=1}^n \frac{\partial \varphi_k}{\partial y_j}(p) \frac{\partial}{\partial x_k}) = \frac{\partial \varphi_i}{\partial y_j}(p).$$

Thus, $\varphi^* dx_i = \sum_{j=1}^m \frac{\partial \varphi_i}{\partial y_j} dy_j$. For a general $k-$form, we can use the fact that pullback preserves wedge products to obtain:

$$\varphi^*\alpha = \sum f_I \circ \varphi \cdot \left(\sum_{i=1}^n \frac{\partial \varphi_{I(1)}}{\partial y_i} dy_i \right) \wedge ... \wedge \left(\sum_{i=1}^n \frac{\partial \varphi_{I(k)}}{\partial y_i} dy_i \right)$$

Exercise 56. Assume $m = n = k$, and that φ is a diffeomorphism. Show that

$$\varphi^*(dx_1 \wedge \cdots \wedge dx_n) = \det(\mathrm{Jac}(\varphi))dy_1 \wedge \cdots dy_n.$$

$\square$

Remark. Assume U, V are bounded.

$$\int_U f dx_1 \cdots dx_n = \int_V (f \circ \varphi)|\det(\mathrm{Jac}(\varphi))|dy_1 \cdots dy_n.$$

If φ is orientation-preserving, that is, $\det(\mathrm{Jac}(\varphi)) > 0$, then we obtain the desired coordinate independence of $\int_U \alpha$.

Exercise 57. Let $\{\varphi_i : V_i \to U_i\}_{i\in I}$ be a smooth atlas for M. Concretely, write what a differential n-form on M is using coordinates. $\square$

17.3. **Directional Derivative.** let M be a smooth manifold. Given $f : M \to \mathbb{R}$, and a tangent vector $v \in T_pM$, we can take directional derivative of f at p in the direction of v:

$$v \cdot f = df_p(v) \in T_{f(p)}\mathbb{R} \simeq \mathbb{R}.$$

If we take a coordinate chart near p with coordinates $x_1, \cdots x_n$, then

$$df_p(v) = (\frac{\partial f}{\partial x_1}(p), \cdots, \frac{\partial f}{\partial x_n}(p))(v_1 \cdots v_n)^T = \sum_{i=1}^n \frac{\partial f}{\partial x}(p) \cdot v_i.$$

Notice that the formula we obtained above coincides with the usual calculus directional derivative.
Remark. Note that on $\mathbb{R}^n$, we have $\left(\frac{\partial}{\partial x_i} \cdot f\right)(p) = \frac{\partial f}{\partial x_i}(p)$.

Definition 54. For a smooth $f : M \to \mathbb{R}$ we define the covector field $df \in \Omega^1(M)$ by

$$df(v) = v \cdot f, \forall p \in M, v \in T_pM.$$

$\square$

Exercise 58. Let $U \subset \mathbb{R}^n$ be open. We already defined $dx_i \in \Omega^1(U)$. Thinking of $x_i : U \to \mathbb{R}$ as a smooth function, justify this notation. Write down df for $f : U \to \mathbb{R}$ in coordinates as before. $\square$

Next time:

- define $d : \Omega^k(M) \to \Omega^{k+1}(M)$
- $\int_M \alpha$, for α differential n-form, M^n oriented compact
- Stokes Theorem: $\int_X d\beta = \int_{\partial X} \beta$.

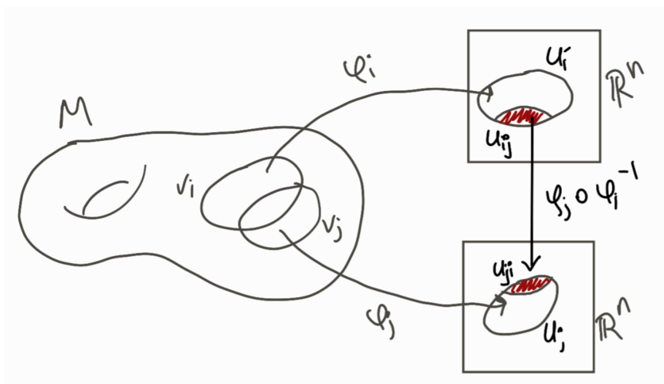
18. April 29, 2022: Differential k-forms on Smooth Manifolds, Exterior Differentiation, De Rham Cohomology

Let M be a smooth manifold and $\{\varphi_i : V_i \to U_i\}_{i\in I}$ a subatlas of the defining maximal smooth atlas. The transition maps are $\varphi_j \circ \varphi_i^{-1} : \phi_i(V_i \cap V_j) \to \phi_j(V_i \cap V_j)$, and we introduce the notation $U_{ij} = \phi_i(V_i \cap V_j) \subset U_i$.

A differential k-form on M is equivalent to the following data,

- $\alpha_i \in \Omega^k(U_i)\ \forall i \in I$ such that,

$$(\varphi_j \circ \varphi_i^{-1})^* \alpha_j|_{U_{ji}} = \alpha_i|_{U_{ij}}$$



Exercise 59. Find a subatlas with two elements for S^2 and write in coordinates what it means to give a differential $0, 1$ and 2-form on S^2.

□

We will now define an operation called exterior differentiation.

$$d : \Omega^k(M) \to \Omega^{k+1}(M)\ ,\ k \geq 0.$$

We have already defined this for $k = 0$,

(0)
$$f \longmapsto df.$$

We also want the following properties,

(1) d is additive,
$$d(\alpha + \beta) = d\alpha + d\beta.$$
(2) $d^2 = 0$ ($d : \Omega^k \to \Omega^{k+1}$ is a coboundary.)
(3) (Leibniz rule) For $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

Proposition 20. *For $U \subset \mathbb{R}^n$ open, there exists a unique $d : \Omega^k(U) \to \Omega^{k+1}(U)$, $k \geq 0$ such that (0), (1), (2) and (3) are satisfied.*

Proof. Recall our notation, $dx_I = dx_{I(1)} \wedge \cdots \wedge dx_{I(k)}$, $I : \{1, \ldots, k\} \to \{1, \ldots, n\}$ and I_k is the set of I which is strictly increasing.

We have to define

$$d(dx_I) = 0$$

$$d(dx_{I(1)} \wedge \cdots \wedge dx_{I(k)}) = \sum_{j=1}^{k} (-1)^{j-1} \cdots \wedge d(dx_{I(j)}) \wedge \cdots = 0$$

For $f : U \to \mathbb{R}$ smooth, we then need

$$d(f dx_I) = df \wedge dx_I$$

We know that every $\alpha \in \Omega^k(U)$ can uniquely be written as $\sum_{I \in I_k} f_I dx_I$. Using (1), we are forced to define for $\alpha = \sum_{I \in I_k} f_I dx_I$,

$$d\alpha = \sum_{I \in I_k} df_I \wedge dx_I$$

What is left is to check that this definition of d indeed satisfies the properties (0), (1), (2) and (3). Property (0) is trivially satisfied, $\alpha = f \implies d\alpha = df$. For (1), first observe that $d(f+g) = df + dg$. Then for $\alpha = \sum_{I \in I_k} f_I dx_I$ and $\beta = \sum_{I \in I_k} g_I dx_I$,

$$\begin{aligned} d(\alpha + \beta) = d\left(\sum_{I \in I_k} (f_I + g_I) dx_I\right) &= \sum_{I \in I_k} d(f_I + g_I) \wedge dx_I \\ &= \sum_{I \in I_k} df_I \wedge dx_I + \sum_{I \in I_k} dg_I \wedge dx_I \\ &= d\alpha + d\beta \ . \end{aligned}$$

Now let us check the generalized Leibniz rule (3) first. By additivity, it suffices to prove for $\alpha = f dx_I$ and $\beta = g dx_J$,

$$d(f dx_I \wedge g dx_J) = ?$$

Start with $k = 0$. Then for every $v \in T_pU$,

$$d(fg)(v) = v \cdot (fg) = f(v \cdot g) + g(v \cdot f) = (f dg + g df)(v)$$

where the second equality follows from the usual Leibniz rule. If confused, note that $v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}$.

$$\begin{aligned} d(\alpha \wedge \beta) = d(f dx_I \wedge g dx_J) &= d(fg dx_I \wedge dx_J) \\ &= d(fg) \wedge dx_I \wedge dx_J \\ &= (fdg + gdf) \wedge dx_I \wedge dx_J \\ &= gdf \wedge dx_I \wedge dx_J + fdg \wedge dx_I \wedge dx_J \\ &= (df \wedge dx_I) \wedge g dx_J + (-1)^k f dx_I \wedge (dg \wedge dx_J) \\ &= d\alpha \wedge \beta + (-1)^k \alpha \wedge \beta \end{aligned}$$

Lastly, we check $d^2\alpha = 0$. Start with $d^2 f \stackrel{?}{=} 0$.

$$d\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i} dx_i\right) = \sum_{i=1}^n d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx_i = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial^2 f}{\partial x^i \partial x^j} dx_j\right) \wedge dx_i = 0$$

Then for $\alpha = \sum_{I \in I_k} f_I dx_I$,

$$\begin{aligned} d^2\alpha = d(d\alpha) &= d\left(\sum_{I \in I_k} df_I \wedge dx_I\right) \\ &= \sum_{I \in I_k} d(df_I \wedge dx_I) \\ &= \sum_{I \in I_k} d^2 f_I \wedge dx_I - \sum_{I \in I_k} df_I \wedge d(1 dx_I) \\ &= 0 \end{aligned}$$

□

Proposition 21. *Let $\varphi : V \to U$ be a diffeomorphism between open subsets of $\mathbb{R}^n$. Then d commutes with φ^*, that is for every $\alpha \in \Omega^k(U)$,*

$$d(\varphi^*\alpha) = \varphi^*(d\alpha) \ .$$

Exercise. Check this for $k = 0$, $f : U \to \mathbb{R}$, $\varphi^* df = d(f \circ \varphi)$ using chain rule.

Proof. Note that $\varphi^* \circ (\varphi^{-1})^* = id$, therefore, φ^* is invertible and its two-sided inverse is $(\varphi^{-1})^*$. We check that

$$\begin{aligned} \Omega^k(U) &\longrightarrow \Omega^{k+1}(U) \ , \ k \geq 0 \\ \alpha &\longmapsto (\varphi^{-1})^* d(\varphi^*\alpha) \end{aligned}$$

satisfies (0), (1), (2), and (3).

(0) $(\varphi^{-1})^* d(\varphi^* f) = d((\varphi^{-1})^* \varphi^*) f = df$ by Exercise.
(1) $(\varphi^{-1})^*$, d, and φ^* is additive.

(2) $(\varphi^{-1})^* d\left(\varphi^*(\varphi^{-1})^* d(\varphi^*\alpha)\right) = 0$

(3) Follows immediately from Leibniz rule for d and that φ^* preserves $\wedge$.

By uniqueness,

$$(\varphi^{-1})^* d(\varphi^*\alpha) = d\alpha \implies d(\varphi^*\alpha) = \varphi^*(d\alpha)$$

$\square$

Finally, we can define

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

Let $\{\varphi_i : V_i \to U_i\}_{i\in I}$ be the defining maximal smooth atlas. Given $\alpha \in \Omega_k(U_i)$ satisfying

$$(\varphi_j \circ \varphi_i^{-1})^* \alpha_j|_{U_{ji}} = \alpha_i|_{U_{ij}}\ ,$$

we can define a differential $k+1$-form by patching together $d\alpha_i \in \Omega^{k+1}(U_i)$, since

$$(\varphi_j \circ \varphi_i^{-1})^*(d\alpha_j|_{U_{ji}}) = d\left((\varphi_j \circ \varphi_i^{-1})^* \alpha_j|_{U_{ji}}\right) = d(\alpha_i|_{U_{ji}}) = d\alpha_i|_{U_{ji}}$$

Exercise. Check (0)-(3) for $d : \Omega^k(M) \to \Omega^{k+1}(M)$, $k \geq 0$ that we have just defined.

18.1. **De Rham Cohomology.** The cohomology of

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{\dim(M)}(M) \xrightarrow{d} 0$$

is defined to be $H^*_{dR}(M)$.

Exercise 60. Check that this agrees with our definition of $H^0_{dR}(U)$ and $H^1_{dR}(U)$ from lectures 1 and 2. $\square$

After defining $\int_M \alpha$, and proving Stokes' theorem, we will follow the outline,

- Step 1: Define a chain map
$$\int : \Omega^k(M) \longrightarrow C^*_{sm}(M;\mathbb{R})$$
- Step 2: Prove that $H^*(\int)$ is an isomorphism.

Exercise 61. Define the map $\int : \Omega^k(M) \longrightarrow C^*_{sm}(M;\mathbb{R})$. What is the big theorem needed to prove that this is a chain map? $\square$

Exercise 62. Read "Smooth Singular Homology", pages 473-480 frın Lee. Summarize its contents and its role in the proof of de Rham theorem (you must have some idea what this theorem will say at this point). $\square$

19. May 9, 2022: Manifolds with Boundary, Integration of Differential Forms

We start by introducing manifolds with boundary. The local models for manifolds with boundary are open subsets of $\mathbb{H}^n$ and $\mathbb{R}^n$, where

$$\mathbb{H}^n := \{(x_1, \dots x_n) | x_n \geq 0\} \subset \mathbb{R}^n,$$

is called the half space. Note that we have $\text{int}(\mathbb{H}^n) = \{x_n > 0\}$ and $\partial\mathbb{H}^n = \{x_n = 0\}$.

Recall that we call a function from a subset of an Euclidean space smooth if it can be extended to a smooth map in an open neighbourhood.

Lemma 5. *Let U and V be open subsets of $\mathbb{H}^n$, if $\phi : U \to V$ is a smooth bijection with a smooth inverse, then it sends $U \cap (\mathbb{R}^{n-1} \times \{0\})$ diffeomorphically to $V \cap (\mathbb{R}^{n-1} \times \{0\})$.*

Proof. It suffices to show $\phi(U \cap (\{0\} \times \mathbb{R}^{n-1})) \subset V \cap (\{0\} \times \mathbb{R}^{n-1})$.

Assume that $\phi(u) = v$ and $u \in \text{int}(U)$. Choose $u_n \to u$ with $\phi(u_n) = v_n \in \text{int}(U)$.

We have $\mathrm{d}\phi^{-1}_{v_n} \circ \mathrm{d}\phi_{u_n} = \text{id}$. Letting $n \to \infty$ and using C^1-ness of ϕ and ϕ^{-1}, $\mathrm{d}\phi_u$ is an isomorphism.

Applying the IFT at u to $\phi|_{\text{int}(u)} : \text{int}(U) \to \mathbb{R}^n$, we find that v has an open neighbourhood $N(V)$ inside $\mathbb{R}^n$ that lies in the image of ϕ and hence inside V, so $v \in \text{int}(V)$.

This implies that if $u \notin \text{int}(U)$, then $\phi(U) \notin \text{int}(V)$ by applying the result we just proved to ϕ^{-1}.

$\square$

Exercise 63. Define a smooth manifold with boundary using atlases. Define the boundary of a smooth manifold with boundary and prove that it is canonically a smooth manifold. $\square$

We can define the following exactly as we did for manifolds:

- Tangent Bundle
- Cotangent Bundle
- Differential Forms
- Exterior Differentiation
- Partitions of Unity Subordinate to an Open Cover Specifically we will need:M smooth manifold with boundary, $V_1, \dots, V_N \subset M$ open sets $\implies \exists \rho_i : M \to \mathbb{R}$ smooth such that:
  (1) $\text{supp}(\rho_i) \subset V_i$
  (2) $\sum_{i=1}^N \rho_i(x) = 1$

- Orientation: A choice of continuously varying orientations of T_xM for all $x \in M$.
  Standart orientation on $\mathbb{H}^n \equiv$ standard orientation on $\mathbb{R}^n$.

19.1. **Integration of Differential Forms.** We call a differential k-form α on a manifold with boundary M compactly supported if $\text{supp}(\alpha) = \overline{\{p \in M | \alpha_p \neq 0\}}$ is compact (or equivalently bounded).

Remark 27. Already for $M = \mathbb{R}^n$ if α is not compactly supported $\int_M \alpha$ may not exist! $\square$

Assume that M is n-dimensional oriented, and $\alpha \in \Omega^n(M)$ is compactly supported (imagine M is compact first).

We will now define $\int_M \alpha \in \mathbb{R}$.

19.2. **Step 1:** Assume that α is supported inside the domain of a connected coordinate chart (could be of either type), for example $\varphi : V \to U$. Then $\alpha = f dx_1 \wedge \ldots \wedge dx_n$ in coordinates, where $f : U \to \mathbb{R}$ is compactly supported.

Define $\int_M \alpha := \pm \int_U f dx_1 \ldots dx_n$, $+$ if the orientation induced on U is the same as the standard one, and $-$ otherwise. We are using ordinary Riemann integral for the right hand side, in particular the symbol $dx_1 \ldots dx_n$ is really a place-holder.

Assume that $\varphi' : V' \to U'$ is another such chart. Let $\psi : \varphi(V \cap V') \to \varphi'(V \cap V')$ be the transition map.

We know that α is supported inside $V \cap V'$ and if $\alpha = f' dx'_1 \wedge \ldots dx'_n$ on U', then $f = (f' \circ \psi)\det(\text{Jac}(\varphi))$.

$$\int_U f = \int_{\varphi(V\cap V')} f = \int_{\varphi(V\cap V')} (f' \circ \varphi)\det(\text{Jac}(\psi)) = \pm \int_{\phi'(V \cap V')} f' = \pm \int_{U'} f'.$$

Then $\int_M \alpha$ is well defined (signs work out exactly as needed).

20. May 12, 2022: Integration of differential forms completed, Stokes theorem (statement)

Let M^n be oriented manifold with boundary (its boundary might be empty) and let $\alpha \in \Omega^n(M)$ be compactly supported. We are trying to define $\int_M \alpha \in \mathbb{R}$.

- Step 1: If α is supported inside the domain of a connected coordinate chart, we did this. If $\alpha = f dx_1 \wedge \cdots \wedge dx_n$ in coordinate charts, then $\int_M \alpha := \pm \int_{\mathbb{R}^n} f$.
  Partitions of Unity: Let $V_1, \ldots, V_N \subset M$ be open subsets, then there exist $\rho_i : M \to \mathbb{R}$ such that

(1) $supp(\rho_i) \subset V_i$
(2) $\sum_{i=1}^{N} \rho_i(x) = 1,\ x \in \cup_{i=1}^{N} V_i$

Remark 28. We can also make some $\rho_i \geq 0$ if we need it. □

Remark 29. Unless M is one dimensional with nonempty boundary, we can find positively oriented subatlas. □

Now, back to the general task.

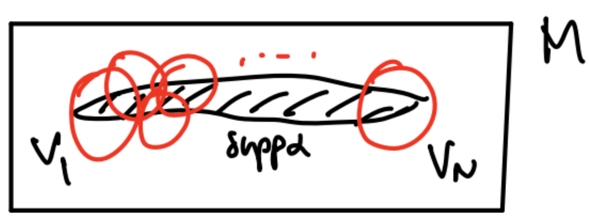


FIGURE 24

- Step 2: (General Case, definition)
  Cover $supp(\alpha)$ with connected domains of coordinate charts $V_1, \ldots, V_N$ with

$$\phi_i : V_i \to U_i$$

Choose a partitions of unity $\rho_1, \ldots, \rho_N$ on $\cup_{i=1}^{N} V_i$ as above.

Define

$$\int_M \alpha := \sum_{i=1}^{N} \int_M \rho_i \alpha$$

- Step 3: (General, well definedness)

Let $\tilde{V_1}, \dots, \tilde{V_n}$ be another cover and $\tilde{\rho_1}, \dots, \tilde{\rho_n}$ another partitions of unity.

$$\begin{aligned}
\sum_{i=1}^{N} \int_M \rho_i \alpha &= \sum_{i=1}^{N} \int_M \rho_i (\sum_{j=1}^{n} \tilde{\rho_j}) \alpha \quad (Why?) \\
&= \sum_{i=1}^{N} \int_M \sum_{j=1}^{n} \rho_i \tilde{\rho_j} \alpha \\
&= \sum_{i=1}^{N} \sum_{j=1}^{n} \int_M \rho_i \tilde{\rho_j} \alpha \\
&= \sum_{j=1}^{n} \sum_{i=1}^{N} \int_M \tilde{\rho_j} \rho_i \alpha \\
&= \sum_{j=1}^{n} \int_M \tilde{\rho_j} \alpha
\end{aligned}$$

Note that in the step from second to third line we changed the order of summation and integration. This is possible because this can be interpreted as doing the same operation for Riemann integrals inside U_i.

This finishes the definition of $\int_M \alpha$. Note that if we change the orientation of M the integral gets negated.

This is a great definition but it is not good for computations.

Proposition 22 (Integration Over Parametrizations)**.** *Let M be a oriented smooth n-manifold with or without boundary, and let w be a compactly supported n-form on M. Suppose $D_1, \dots, D_k$ are open domains of integration in $\mathbb{R}^n$, and for $i = 1, \dots, k$, we are given smooth maps $F_i : \overline{D_i} \to M$ satisfying*

1. *F_i restricts to an orientation-preserving diffeomorphism from D_i onto an open subset $W_i \subseteq M$:*
2. *$W_i \cap W_j = \emptyset$ when $i = j$:*
3. *$supp(w) \subseteq \overline{W_1} \cup \cdots \cup \overline{W_k}$.*

Then

$$\int_M w = \sum_{i=1}^{k} \int_{D_i} F_i^* w$$

This is a statement directly taken from Lee. Domain of integration means bounded and topological boundary is measure 0.

Exercise 64. Define a nowhere vanishing differential two form α on S^2. Use it to orient S^2. Compute $\int_{S^2} \alpha$.
Directly show that there is no β such that $d\beta = \alpha$ □

20.1. **Boundary orientation.** M^n an oriented manifold with boundary, then ∂M is canonically a smooth manifold (without boundary).

Claim 1. *∂M is canonically oriented.*

Proof. Let $x \in \partial M$ and $V_1, \dots, V_{n-1}$ be a basis of $T_x \partial M$.

We declare $V_1, \dots, V_{n-1}$ to be positive if $N, V_1, \dots, V_{n-1}$ is positive for $T_x M$, where $N \in T_x M$ is a strictly outward pointing vector in some chart $\phi : V \to U \subset \mathbb{H}^n$.

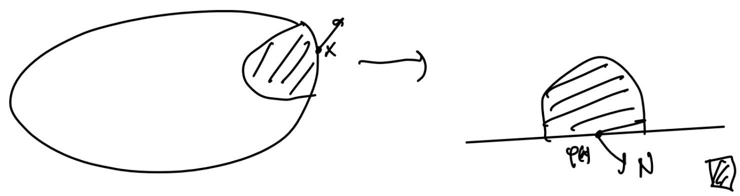


FIGURE 25

□

Exercise 65. Prove that this indeed defines an orientation. □

Theorem 10 (Stokes Theorem)**.** *Let M^n be an oriented manifold with boundary, ∂M be oriented as above and $\alpha \in \Omega^{n-1}(M)$ compactly supported.*

$$\int_M d\alpha = \int_{\partial M} i^*_{\partial M} \alpha$$

Exercise 66. Carefully write down the four famous special cases from Calculus of this theorem. Start with notations used in Calculus and translate them into integrals of differential forms. □

Corollary 3. *Let M be a manifold with boundary, $X^k \subset M$ be a compact and oriented submanifold (without boundary). Let $\alpha \in \Omega^k(M)$ closed ($d\alpha = 0$) and assume*

$$\int_S i^*_X \alpha \neq 0.$$

Then

(1) *X does not bound a $(k+1)$ dimensional submanifold with boundary.*

(2) *α is not exact ($d\beta$, for some $\beta \in \Omega^{k-1}(M)$)*

□

Corollary 4. *If M^n is a connected, closed (compact without boundary) oriented manifold and $\alpha \in \Omega^n(M)$ is nowhere vanishing, then α is not exact.* □

Proof. Let's prove that $\int_M \alpha \neq 0$ which finishes the proof by Stokes theorem.

Let us cover M by domains of positively oriented connected coordinate charts $\phi : U_i \to V_i$ and choose partitions of unity $\rho_1, \ldots, \rho_N$ with $\rho_i \geq 0$.

Write $\alpha = f_i dx_1 \wedge \cdots \wedge dx_n$, $i = 1, \ldots, N$.

Each f_i is either everywhere positive or everyhwhere negative on U.

We claim that all f_i's have the same sign. If $V_i \cap V_j \neq 0$, then f_i and f_j have the same sign as both charts are positively oriented. We finish by connectedness. Then

$$\int_M \alpha = \sum_{i=1}^{N} \int_{\mathbb{R}^{\ltimes}} \rho_i f_i \neq 0$$

□

We will continue with the following proposition next time.

Proposition 23. *Every oriented manifold M^n with boundary admits a nowhere vanishing differential n-form.*

21. May 16, 2022: Volume forms, Poincare duality in deRham theory, proof of Stokes theorem

Let us start with the proof of the proposition from the end of last lecture.

Proof. M admits a Riemannian metric g. For every $x \in M$, there is a canonical element

$$vol_{g,x} \in \Lambda^k T^* M$$

such that if $\{e_1, \cdots, e_n\}$ is an oriented orthonormal basis of $T_x M$ then

$$vol_{g,x}(e_1, \cdots, e_n) = 1.$$

Exercise. Prove that this defines a nowhere vanishing differential n-form. Explain how it is related to ds, dA, dV (line, surface/area, and volume) elements from Calculus.

□

Corollary 5. *If M^n is closed oriented manifold, then $H^n_{dR}(M) \neq 0$.* □

Theorem 11. *If M^n closed oriented connected manifold. Then, we have the isomorphism*

$$H^n_{dR}(M) \xrightarrow{\sim} \mathbb{R}$$
$$[\alpha] \mapsto \int_M \alpha$$

This is equivalent to showing that if $\int_M \alpha = \int_M \beta$, then α and β are cohomologous. Which in turn is equivalent to

$$\int_M \alpha = 0 \Leftrightarrow \alpha \text{ is exact.}$$

As far as I know this does not have a simple proof.

Remark 30. There is a proof in Guillemin-Haine for going from a slightly generalized version of this statement (using compactly supported forms) in the special case $\mathbb{R}^n$ to the general case and I do not understand it. It is in the section "Degree theory on manifolds." □

This theorem is a special case of Poincaré duality. We will cover this next year but here is an overview.

Let M^n be an oriented smooth manifold without boundary. We can define $H^*_{dR,c}(M)$ by considering compactly supported differential forms:

$$H^*(\Omega^0_{cp}(M) \xrightarrow{d} \Omega^1_{cp}(M) \to \cdots)$$

Here is the main statement

Theorem 12. *There is a perfect pairing*

$$H^k_{dR}(M) \times H^{n-k}_{dR,c}(M) \to \mathbb{R},$$
$$([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta.$$

Corollary 6. $H^k_{dR}(M) \cong (H^{n-k}_{dR,c}(M))^V$.

If M is closed, remove c. □

The canonical proof of this is very similar to the proof of deRham theorem that we will give.

For M closed, there is also an approach called "Hodge theory" (in the historical account of Samelson from the website attributed to Volterra).

In this approach, one fixes a Riemannian metric g and finds canonical (harmonic, notion depends on g) representatives of deRham cohomology classes. These are differential forms which satisfy $d\alpha = 0$ and $d^*\omega = 0$. Equivalently: for $\Delta : dd^* + d^*d$ "Laplacian", $\Delta\alpha = 0$.

Harmonic differential forms are closed under multiplication by an operator called Hodge star

$$* : \Omega^k(M) \to \Omega^{n-k}(M).$$

Concretely, $\Delta\alpha = 0 \implies \Delta * \alpha = 0$

Moreover, $\int_M \alpha \wedge *\alpha \neq 0$ ($\alpha \neq 0$) implies the Poincaré Duality.

Finally, note that if $X^k \subset M$ closed submanifold, then we obtain a well defined element in $(H^k_{dR}(M))^V$:

$$[\alpha] \mapsto \int_X i_x\alpha$$

Under Poincaré duality, this must be given by some class in $H^{n-k}_{dR,c}(M)$. One can construct a representation of this class as a Dirac delta form located at X - Thom forms. This leads us to an approach to Intersection Theory of submanifolds using differential forms.

Back to proof of Stokes' theorem:

Theorem 13. *Let M^n be an oriented manifold with boundary and $\alpha \in \Omega^{n-1}(M)$ compactly supported.*

$$\int_M d\alpha = \int_{\partial M} i^*_{\partial M}\alpha$$

Proof. We start with the case $M^n = \mathbb{H}^n$. We can write

$$\alpha = \sum_{i=1}^n f_i dx_1 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n$$

$$\implies d\alpha = \sum_{i=1}^n (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n$$

$$\implies \int_M d\alpha = \sum_{i=1}^n (-1)^{i-1} \int_{\mathbb{H}^n} \frac{\partial f_i}{\partial x_i}(dx_1 \ldots dx_n).$$

For the other side of Stokes theorem, we have

$$\int_{\partial M} i^*\alpha = (-1)^n \int_{\mathbb{R}^{n-1}\times\{0\}} f_n(dx_1 \ldots dx_{n-1}) \tag{2}$$

Exercise 67. Write a full proof of (2). ∎

Exercise 68. Finish the proof of $M = \mathbb{H}^n$ case. **Hint:** You only need Fubini's theorem on integrals and Fundamental theorem of Calculus.
$\square$

The proof of $M = \mathbb{R}^n$ is even easier and also left as an exercise.

Let us move on to the general case. Choose a partition of unity $\rho_1, \ldots, \rho_N$ which can be used in the definition of $\int_M d\alpha$. Then $\rho_1\big|_{\partial M}, \ldots, \rho_N\big|_{\partial M}$ can be used to define $\int_{\partial M} i^*\alpha$.

$$\int_M d\alpha = \sum_{i=1}^N \int_M \rho_i d\alpha$$

$$\begin{aligned}\int_{\partial M} i^*\alpha &= \sum_{i=1}^N \int_{\partial M} \rho_i\big|_{\partial M} i^*\alpha \\ &= \sum_{i=1}^N \int_M d(\rho_i\alpha)\end{aligned}$$

Exercise. Justify this step using the $M = \mathbb{H}^n$ and $M = \mathbb{R}^n$ cases.

Using Leibniz rule

$$\begin{aligned}\int_{\partial M} \alpha &= \sum_{i=1}^N \left(\int_M \rho_i d\alpha + \int_M d\rho_i \wedge \alpha\right) \\ &= \int_M d\alpha + \int \left(\sum_{i=1}^N d\rho_i\right) \wedge \alpha.\end{aligned}$$

Finally, note that the second summand is 0 since $d(1) = 0$. $\square$

Exercise. Justify the commutativity of integral and summation that we just did.

22. May 22, 2022: Proof of de Rham Theorem, part 1

Let M be a smooth manifold without boundary. Let us identify $\Delta^k \subset \mathbb{R}^{k+1}$ with its image in $\mathbb{R}^k$ under the projection to last k coordinates.

Orient $\Delta^k \subset \mathbb{R}^k$ in the standard way. We can define a linear map

$$\int : \Omega^k(M) \to C^k_{sm}(M; \mathbb{R})$$

$$\alpha \mapsto \left((\sigma : \Delta^k \to M) \mapsto \int_{\Delta^k} \sigma^*\alpha\right)$$

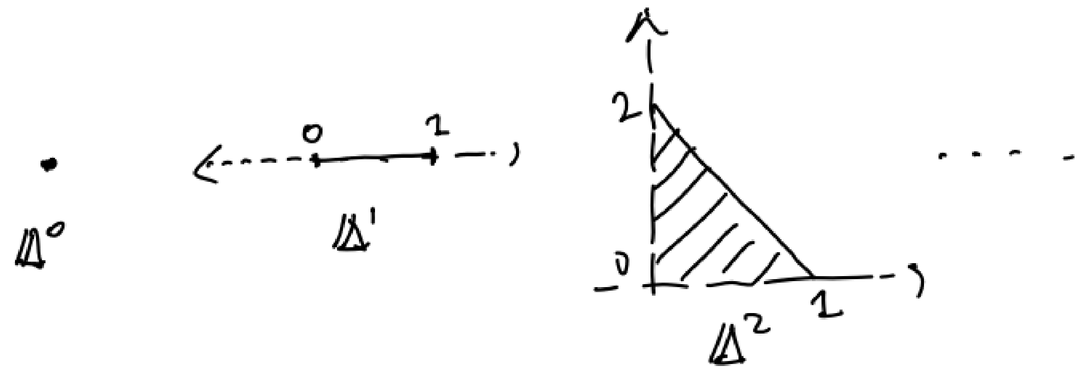


FIGURE 26. Projection to last k coordinates

Claim: This is a chain map.

Proof. We need to show

$$d\alpha \mapsto \left((\gamma : \Delta^{k+1} \to M) \mapsto \sum_{i=0}^{k+1} (-1)^i \int_{\Delta^k} (\gamma \circ \mathrm{face}_i)^* \alpha \right) .$$

In other words

$$\int_{\Delta^{k+1}} \gamma^* d\alpha = \sum_{i=0}^{k+1} (-1)^i \int_{\Delta^k} (\gamma \circ \mathrm{face}_i)^* \alpha$$

Using Stokes theorem for the LHS, and working out the signs for the RHS we can show that both are equal to

$$\int_{\partial \Delta^{k+1}} \left(\gamma|_{\partial \Delta^{k+1}}\right)^* \alpha .$$

□

Theorem 14. ***(de Rham)**: $\int : \Omega^k(M) \to C^*_{sm}(M;\mathbb{R})$ induces an isomorphism on homology.*

If for M this is true, let us call it a **good** manifold. We want to prove that all manifolds are good. Note that empty set is good. It is easy to see that goodness is preserved under diffeomorphisms.

Let us define a good cover $M = \bigcup_{i \in I} U_i$ by open subsets as a cover where each finite intersection $\bigcap_{j \in J} U_j$ is good. Additionally, if $\{U_i\}_{i \in I}$ is a basis of the topology we call $\{U_i\}_{i \in I}$ a good basis.

Here are the main steps:

(1) All open boxes

$$(a_1, b_1) \times \ldots \times (a_n, b_n) \subset \mathbb{R}^n$$

are good.

(2) If M has a finite good cover, then it is good.

There are two more steps:

(3) If M_i is good for $i \in N$, then $\bigsqcup M_i$ is good.

(4) If M has a good basis, then it is good.

We end with:

(5) If $U \subset \mathbb{R}^n$ is open, U is good.

(6) All smooth manifolds are good.

Exercise 69. Deduce (5) and then (6) from (1)-(4). □

We will focus on (1) and (2). (3) is about following definitions correctly and figuring out correctly what is a direct sum and what is a direct product. (4) relies on the fact that M admits on exhaustion by compact domains,

$$K_1 \subset K_2 \subset \ldots \subset M \text{ s.t}$$
$$\bigcup K_i = M \text{ and } K_i \subset \text{int}(K_{i+1})$$

and uses (2) and (3).

The proof is short and cool. Please think about it and check with Lee. (2) relies on the Mayer-Vietoris property for both deRham and singular cohomology (next time). Today we focus on (1).

Lemma 6 (Poincaré Lemma). *$H^k_{dR}(\mathbb{R}^n) = 0$ for $k > 0$. Please check that this implies* (1)*.*

Proof. We prove this by induction on dimension n. It is true for $n = 0$.

Let us define a $h : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$ as follows: for $\alpha = \sum_{I \in I_k} f_I dx_I$

$$h\alpha_x := \sum_{I \in I_k, I(1)=1} (\int_0^{x_1} f_I) dx_{I(1)} \wedge \ldots \wedge dx_{I(k)}$$

$$dh\alpha = \sum_{I \in I_k, I(1)=1} f_I dx_{I(1)} \wedge \ldots \wedge dx_{I(k)} + \sum_{I \in I_k, I(1)=1, j>0} \frac{\partial}{\partial x_j} (\int_0^{x_1} f_I) dx_j \wedge dx_{I(2)} \ldots \wedge dx_{I(k)}$$

$$hd\alpha = \sum_{I \in I_k, I(1)>1} (\int_0^{x_1} \frac{\partial f_I}{\partial x_1}) dx_I - \sum_{I \in I_k, I(1)>1, j>1} (\int_0^{x_1} \frac{\partial f_I}{\partial x_j}) dx_j \wedge dx_{I(2)} \ldots \wedge dx_{I(k)}$$

$$\implies dh\alpha - hd\alpha = \alpha - \sum_{I \in I_k, I(1)>1} f_I(0, x_2, .., x_n) dx_I$$

Writing this more concisely,

$$dh\alpha - hd\alpha = \alpha - \pi^* \iota^* \alpha,$$

where $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is the inclusion by setting the first coordinate to 0 and $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection to the last $n-1$ coordinates.

If α is closed,

$$d(h\alpha) = \alpha - \pi^*(\text{closed } k\text{-form in } \mathbb{R}^{n-1})$$

We finish using the induction hypothesis. □

The larger context for this statement is homotopy invariance maps induced on deRham cohomology (similar to what we discussed for singular homology). The main point in establishing this is Cartan's magic formula $\mathcal{L}_v \omega = \iota_v d\omega + d\iota_v \omega$ (next year).

23. May 23, 2022: Proof of de Rham Theorem, part 2

We start with some homological algebra.

Definition 55. A short exact sequence of a vector space is a diagram of two linear maps defined as

$$V \xrightarrow{f} V'' \xrightarrow{g} V'$$

such that

(1) f is injective,
(2) $\ker(g) = im(f)$,
(3) g is surjective.

□

Definition 56. A short exact sequnce of cochain complexes is two chain maps

$$A \xrightarrow{F} B \xrightarrow{G} C$$

such that for all $n \in \mathbb{Z}$

$$A^n \xrightarrow{F^n} B^n \xrightarrow{G^n} C^n$$

is a short exact sequence. □

A short exact sequence of cochain complexes look like this:

$$\begin{tikzcd}
 & \cdot \arrow[d, "\delta_{n-1}"] & \cdot \arrow[d, "\delta_{n-1}"] & \cdot \arrow[d, "\delta_{n-1}"] & \\
0 \arrow[r] & A^n \arrow[r, "F^n"] \arrow[d, "\delta_n"] & B^n \arrow[r, "G^n"] \arrow[d, "\delta_n"] & C^n \arrow[r] \arrow[d, "\delta_n"] & 0 \\
0 \arrow[r] & A^{n+1} \arrow[r, "F^{n+1}"] \arrow[d, "\delta_{n+1}"] & B^{n+1} \arrow[r, "G^{n+1}"] \arrow[d, "\delta_{n+1}"] & C^{n+1} \arrow[r] \arrow[d, "\delta_{n+1}"] & 0 \\
 & \cdot & \cdot & \cdot &
\end{tikzcd}$$

All squares are commutative.

Lemma 7. *A short exact sequence of cochain complexes $A \to B \to C$ give rise to a long exact sequence:*

$$\begin{tikzcd}
 & \cdots & \\
H^{n+1}(A) \arrow[r] & H^{n+1}(B) \arrow[r] & H^{n+1}(C) \arrow[ull, dashed] \\
H^n(A) \arrow[r] & H^n(B) \arrow[r] & H^n(C) \arrow[ull, dashed] \\
 & \cdots \arrow[ull, dashed] &
\end{tikzcd}$$

where

$$c^n : H^n(C) \longrightarrow H^{n+1}(A)$$

are canonical maps called connecting maps.

Proof. Take $\alpha \in C^n$ cocyle ($\delta\alpha = 0$). Find $\beta \in B^n$ such that $G_n\beta = \alpha$. We have $G^{n+1}\delta\beta = \delta G^n\beta = \delta\alpha = 0 \Rightarrow \exists! \gamma \in A^{n+1}$ such that $F^{n+1}\gamma = \beta$. We set $c^n([\alpha]) = [\gamma]$. □

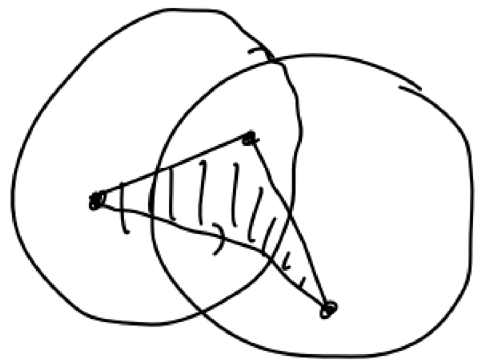
Exercise. Check that $\delta\gamma = 0$, $[\gamma]$ is independent of choices and the resulting sequence is a long exact sequence.

We now construct the Mayer-Vietoris sequence for the deRham cohomology.

Theorem 15. *M smooth manifold. $U, V \in M$ open subsets. Then*

$$\begin{aligned}
\Omega^*(U \cup V) \to \Omega^k(U) \oplus \Omega^k(V) &\to \Omega^k(U \cap V) \\
\alpha \mapsto (\iota_U^*\alpha, \iota_V^*\alpha) & \\
(\beta_U, \beta_V) &\mapsto \iota^*\beta_U - \iota^*\beta_V
\end{aligned}$$

is a short exact sequence of cochain complexes. Therefore we obtain the long exact sequence e.g. send



to 1

the rest to 0.

FIGURE 27. $f(\sigma) = 1$ otherwise 0

$$\begin{array}{ccccc} & & \cdots & & \\ \cdots \to H^n_{dR}(U \cup V) & \longrightarrow & H^n_{dR}(U) \oplus H^n_{dR}(V) & \longrightarrow & H^n_{dR}(U \cap V) \\ & & \cdots & \longrightarrow & H^{n+1}_{dR}(U \cap V) \end{array}$$

Proof. Injectivity and the middle statement is easy to prove. It suffices to show surjectivity. Let $\beta \in \Omega^n(U \cap V)$. We need to find $\beta_U \in \Omega(U), \beta_V \in \Omega^n(V)$ such that $\beta = \iota^*\beta_U - \iota^*\beta_V$. Choose ρ_U, ρ_V

(1) $\rho_U + \rho_V = 1$ on $U \cup V$
(2) $supp(\rho_U) \subset U, supp(\rho_V) \subset V$

$$\beta_{U,x} := \begin{cases} \rho_V(x)\beta, & x \in U \cap V \\ 0, & x \notin \operatorname{supp}(\rho_V) \end{cases}, \quad \beta_{V,x} := \begin{cases} -\rho_U(x)\beta, & x \in U \cap V \\ 0, & x \notin \operatorname{supp}(\rho_u) \end{cases}$$

These are easily seen to be smooth and we have $\beta = \iota^*\beta_U - \iota^*\beta_V$. $\square$

Let us also briefly discuss the Mayer-Vietoris for singular cohomology. Is the following

$$C^*(U \cup V) \to C^*(U) \oplus C^*(V) \to C^*(U \cap V)$$

a short exact sequence? Here surjectivity and middle slot is fine, but injectivity is actually wrong! See Figure 27.

Let's focus on the Mayer-Vietoris for singular homology. We have an obvious exact sequence.

$$C_*(U \cup V) \to C_*(U) \oplus C_*(V) \to C_*(U) + C_*(V)$$

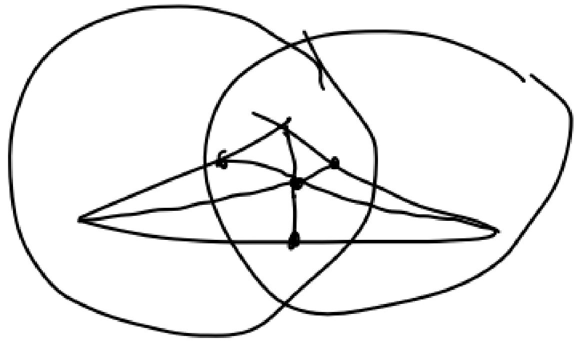


FIGURE 28. Baryocentric Subdivision Operator

Then, using bary-centric subdivision, we prove that $C_*(U) + C_*(V) \subset C_*(U \cup V)$ induces isomorphism on homology. This takes serious effort (see Hatcher for example for proof.)

Upshot for us: There exists canonical Mayer-Vietoris exact sequence.

$$\begin{tikzcd}
{} \arrow[r, dashed] & H^{n+1}(U \cup V) & H^{n+1}(U) \oplus H^n(V) & H^{n+1}(U \cap V) \\
{} \arrow[r, dashed] & H^n(U \cup V) \arrow[r] & H^n(U) \oplus H^n(V) \arrow[r] & H^n(U \cap V) \\
& & \cdots &
\end{tikzcd}$$

where the connecting maps $H^n(U \cap V) \to H^{n+1}(U \cup V)$ do the following:

Take $f \in H^n(U \cap V)$. We need to define $c_n f(\sigma) \in \mathbb{R}$ for a singular cycle $\sigma \in C_{n+1}(U \cup V)$ such that if σ is a boundary, we get zero.

If $\sigma = \sigma_U + \sigma_V +$ boundary, we know that $\partial\sigma_U + \partial\sigma_V = O$. This means that $[\partial\sigma_U]$ comes from a class in $H_n(U \cap V)$ $[\partial\sigma_U = -\partial\sigma_V \to$ all summands of $\partial\sigma_U$ needs to have images contained in V and hence $U \cap V]$. Define $c_n f(\sigma) = f([\partial\sigma_U])$.

Exercise. Check independence under choices.

Proposition 24. *The diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H^n_{dR}(U) \oplus H^n_{dR}(V) \arrow[r] \arrow[d, "H(\int)"] & H^n_{dR}(U \cap V) \arrow[r] \arrow[d, "H(\int)"] & H^{n+1}_{dR}(U \cup V) \arrow[r] \arrow[d, "H(\int)"] & \cdots \\
\cdots \arrow[r] & H^n(U) \oplus H^n(V) \arrow[r] & H^n(U \cap V) \arrow[r] & H^{n+1}(U \cup V) \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

Proof. We focus only on the rightmost square of the diagram. The rest is easy.

$c_n[\alpha] = [\textit{patch together } d\alpha_U \textit{ and } d\alpha_V \textit{ to a closed form on } U \cup V] = [\beta]$

$\alpha = \iota^*\alpha_U - \iota^*\alpha_V$

$\Rightarrow d\alpha_U$ agrees with $d\alpha_V$ on $U \cap V$.

We also write $\sigma = \sigma_U + \sigma_V +$ boundary.

Below the expression $\int \sigma^*\beta$ means $\int \beta$ applied to σ.

$$\int \sigma^*\beta = \int \sigma_U^*\beta + \int \sigma_V^*\beta = \int \partial\sigma_U^*\alpha_U + \int \partial\sigma_V^*\alpha_V \tag{3}$$

$$= \int \partial\sigma_U^*\alpha_U - \int \partial\sigma_U^*\alpha_V = \int \partial\sigma_U^*\alpha \tag{4}$$

□

Now we finish the proof of finite good cover $\Rightarrow$ good.

Let us consider $M = U \cup V$ first.

$$\begin{tikzcd}
\cdots \arrow[r] & H^n_{dR}(U) \oplus H^n_{dR}(V) \arrow[r] \arrow[d, "\cong"] & H^n_{dR}(U \cap V) \arrow[r] \arrow[d, "\cong"] & H^{n+1}_{dR}(U \cup V) \arrow[r] \arrow[d, "?"] & \cdots \\
\cdots \arrow[r] & H^n(U) \oplus H^n(V) \arrow[r] & H^n(U \cap V) \arrow[r] & H^{n+1}(U \cup V) \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 70. Prove by 'diagram chasing' that the remaining arrows have to be isomorphism as well. □

Remark 31. This is called the 5 Lemma. □

The general case $M = \bigcup_{i=1}^N U_i$ is handled by induction on N.

Note $\left(\bigcup_{i=1}^{N-1} U_i\right) \cap U_N = \bigcup_{i=1}^{N-1}(U_i \cap U_N)$ is a good cover as well!

$$\Rightarrow \left(\bigcup_{i=1}^{N-1} U_i\right) \cap U_N \text{ is good.}$$

Exercise 71. Apply the same argument with the Mayer-Vietoris sequences for $U = \bigcup_{i=1}^{N-1} U_i$ and $V = U_N$ and finish the proof. □