Lecture 1

Overview and Fukaya categories of surfaces

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1 Overview

The goal of these lectures is to prove homological mirror symmetry (HMS) in some examples, using the methods of *Fukaya categories with support.* We have three settings in mind:

(i) **Affine:** a smooth affine log Calabi–Yau variety X over $\mathbb{C}$.
(ii) **Compact:** a compact Calabi–Yau manifold M with a Kähler class.
(iii) **Projective:** the special case in which M is projective.

In the affine setting we use an exact symplectic structure coming from an embedding into $\mathbb{C}^n$. The expected equivalences have the form

$$\begin{aligned} D^\pi \mathcal{W}(X) &\simeq D^b \operatorname{Coh}(X^\vee), && \text{affine case,} \\ D^\pi \operatorname{Fuk}(M) &\simeq D^b \operatorname{Coh}(M^\vee), && \text{compact case.} \end{aligned}$$

Here the left-hand side is the A-side and the right-hand side is the B-side. The notation D^π means that we take twisted complexes and split idempotents. We understand the equivalences as exact equivalences of triangulated categories, but they can be lifted to A_∞ or dg enhancements.

In the examples we aim to treat, the mirrors have the following form:

Affine case	$X^\vee$: a nonproper scheme smooth over $\mathbb{Z}$;
Compact case	$M^\vee$: a smooth proper rigid analytic space over $\Lambda_{\mathbb{Q}}$;
Projective case	$M^\vee$: a smooth projective scheme over $\Lambda_{\mathbb{Q}}$.

In particular, the mirror of an affine variety is often *not affine.* These descriptions are the target of the construction in the examples, rather than a general existence assertion for every Calabi–Yau manifold.

Our strategy is inspired by the SYZ proposal. For the purposes of these lectures, we take this to mean that the A-side admits a Lagrangian torus fibration whose regular fibers have Maslov class zero, with a finite list of allowed local models. These local models will be full subdomains of special examples in the affine setting. Each has a local mirror, and the global mirror is obtained by gluing these local mirrors, taking into account *quantum corrections.*

We will use integral or Novikov coefficients on the B-side, rather than complex coefficients. We also equip our geometric data with a *reference Lagrangian section R.* This section will be mirror to the structure sheaf in a strong sense, by construction. Among the affine examples, the local models are singled out by having an *affine* HMS mirror. More details will come in later lectures. We will also discuss enhancing the A-side with large volume limit data.

The first lecture is organized around the following topics:

1. Lagrangian Floer theory and the Fukaya pre-A_∞ category of a graded surface with an area form;
2. the wrapped category of T^*S^1;
3. HMS for T^*S^1 and for the two-dimensional torus.

We begin with the surface construction, following Abouzaid [1] and de Silva–Robbin–Salamon [2], and then turn to the wrapped cotangent cylinder, one-dimensional mirrors, and the two HMS statements.

2 Graded curves on a surface

Let (Σ, ω) be a surface with an area form. We fix an *oriented line field* η on Σ. Its underlying unoriented line field supplies the grading structure; its orientation will let us obtain orientations of curves from their gradings.

2.1 The curves

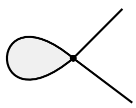
A curve will be specified by a proper immersion

$$\iota : \widetilde{\gamma} \longrightarrow \Sigma,$$

where $\widetilde{\gamma}$ is a finite disjoint union of circles and copies of $\mathbb{R}$, and self-intersections are transverse double points. Every such curve is Lagrangian, since its dimension is one. We impose the following additional conditions on the curves used as objects:

- a circle component is not null-homotopic;
- there are no fishtails.

A fishtail is an immersed disk with one boundary corner, with boundary on the curve and with a continuous lift of the boundary away from that corner to $\widetilde{\gamma}$. Thus one excludes a contractible teardrop made by a single branch of the curve. Abouzaid's conditions (noncontractible circle components and no fishtails) are equivalent to requiring that each component, lifted from its universal covering line to the universal cover of Σ, be a properly embedded line. This is the unobstructed class of curves used here; we do not introduce bounding cochains.



excluded fishtail

2.2 Gradings and orientations

Write $\ell_\gamma(p) = d\iota_p(T_p\widetilde{\gamma})$ for the tangent line of the chosen branch at p. A grading is a continuously varying homotopy class, relative to the endpoints, of paths

$$h_\gamma(p) : \ell_\gamma(p) \rightsquigarrow \eta^{\mathrm{unor}}_{\iota(p)} \quad \text{in } \mathbb{P}(T_{\iota(p)}\Sigma).$$

The use of $p \in \widetilde{\gamma}$ matters: at a self-intersection there are two tangent lines and two grading values.

Starting with the chosen orientation of η, lift the reversed path $h_\gamma(p)^{-1}$ to the double cover of oriented lines. Its endpoint gives an orientation of $\ell_\gamma(p)$, and hence of $\widetilde{\gamma}$. Thus

$$\text{grading} \quad \Longrightarrow \quad \text{orientation.}$$

Changing the grading by one reverses this orientation. If one omits the orientation of η, the integer grading construction still makes sense, but the orientations needed for signs must be supplied separately.

Existence of an oriented line field is an assumption on the surface. Among closed connected oriented surfaces, it exists only on the torus. Abouzaid's construction on closed higher-genus surfaces is $\mathbb{Z}/2$-graded; here we use its integer-graded version on surfaces carrying the stated grading data.

2.3 Degrees at intersections

Let α, β be graded curves meeting transversely at finitely many simple points. For $x \in \alpha \cap \beta$, concatenate

$$T_x\alpha \xrightarrow{h_\alpha} \eta_x^{\text{unor}} \xrightarrow{h_\beta^{-1}} T_x\beta \xrightarrow{\text{short counterclockwise path}} T_x\alpha. \tag{1}$$

Its winding number in $\mathbb{P}(T_x\Sigma) \cong \mathbb{R}/\mathbb{Z}$ is the degree $|x|$. A counterclockwise rotation through π has winding number 1. "Short" means that the rotation angle lies in $(0, \pi)$.

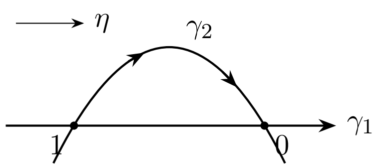
Concretely, choose local positively oriented coordinates with η horizontal. Let $a_\alpha, a_\beta \in \mathbb{R}$ be the lifted tangent angles, measured in units of π, obtained by reversing the grading paths. Then

$$|x| = \lceil a_\beta(x) - a_\alpha(x) \rceil. \tag{2}$$

In particular, the two degrees at the same intersection, viewed as a morphism in opposite directions, satisfy

$$|x|_{\alpha,\beta} + |x|_{\beta,\alpha} = 1.$$

Modulo 2, the degree is 1 precisely when the ordered pair of induced orientations $(T_x\alpha, T_x\beta)$ agrees with the orientation of Σ.



For the displayed lifts of the tangent angles, the left intersection has degree 1 and the right one has degree 0.

3 Coefficients, exactness, and brane data

3.1 Novikov coefficients

For a coefficient ring K, write

$$\Lambda_K = \left\{ \sum_{\lambda \in \mathbb{R}} a_\lambda T^\lambda : a_\lambda \in K, \quad \#\{\lambda \le C : a_\lambda \neq 0\} < \infty \text{ for every } C \in \mathbb{R} \right\}.$$

The valuation of a nonzero series is its least exponent. We will use $\Lambda_{\mathbb{Z}}$ for the surface construction; $\Lambda_{\mathbb{Q}}$ is the Novikov field appearing in the overview. The powers of T record symplectic areas.

Without local systems, the morphism module of a transverse pair is

$$CF^*(\alpha, \beta) = \bigoplus_{x \in \alpha \cap \beta} \Lambda_{\mathbb{Z}}\, x, \qquad \deg(x) = |x|.$$

An intersection point includes its chosen preimage on each normalization. For simple intersections these preimages are implicit.

3.2 Exact and strongly exact curves

Suppose $\omega = d\theta$. The immersion ι is *exact* if

$$\iota^*\theta = df$$

for a function f on $\widetilde{\gamma}$. It is *strongly exact* if f can be chosen so that

$$f(p_1) = f(p_2) \quad \text{whenever} \quad \iota(p_1) = \iota(p_2).$$

Ordinary exactness is the condition needed for the exact coefficient version of the polygon construction; strong exactness is an additional condition, not part of the definition of an exact immersed curve, but is another way to exclude fishtails.

For exact curves one can count polygons over $\mathbb{Z}$, omitting their area weights. The reason is that Stokes' theorem determines the area from the corners and the chosen primitives. We spell this out below.

3.3 Marked points, spin structures, and local systems

Each circle component is equipped with a marked point, away from all intersections in a transverse tuple. To make the sign convention explicit, use the nonbounding spin structure as a reference. Put a label $\sigma \in \{0, 1\}$ at the marked point:

$$\sigma = 0 : \text{ nonbounding spin structure}, \qquad \sigma = 1 : \text{ bounding spin structure}.$$

Each passage through a marked point contributes $(-1)^\sigma$. Thus taking $\sigma = 1$ on every circle gives the marked-point convention used by Abouzaid. Taking $\sigma = 0$ gives a convenient alternative when local systems are included. On a line there is a unique spin structure up to isomorphism and no spin marked point is needed.

We may also equip $\widetilde{\gamma}$ with a rank-one local system E. In the exact integral version its circle holonomy lies in $\mathbb{Z}^\times = \{\pm 1\}$. In the Novikov version we take holonomy in

$$\Lambda_{\mathbb{Z}}^{\times,0} := \{u \in \Lambda_{\mathbb{Z}}^\times : \mathrm{val}(u) = 0\}.$$

Explicitly these units have leading term $+1$ or -1 and all other terms of positive valuation. These are the unitary local systems we use in the Novikov version.

The morphism module is now

$$CF^*((\alpha, E_\alpha), (\beta, E_\beta)) = \bigoplus_{x \in \alpha \cap \beta} \mathrm{Hom}_K((E_\alpha)_x, (E_\beta)_x),$$

with the summand at x placed in degree $|x|$; here K denotes the chosen coefficient ring, either $\mathbb{Z}$ or $\Lambda_{\mathbb{Z}}$.

One can trivialize a circle local system away from its marked point. Crossing the cut in the positive direction contributes its holonomy h; crossing in the negative direction contributes h^{-1}. Spin signs count crossings modulo 2, whereas holonomy records the direction as well. Changing the spin structure has the same effect on polygon weights as tensoring the local system with the sign local system of holonomy -1.

4 The polygon operations

A finite ordered tuple $(L_0, \dots, L_d)$ is *transverse* if its curves are pairwise transverse with finitely many intersections, there are no triple branch intersections, and none of these intersections lies at a marked point. Intersections between distinct curves also avoid their self-intersection points. All subsequences of a transverse tuple are transverse.

4.1 The polygons being counted

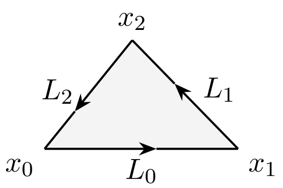
Choose input intersections $x_i \in L_{i-1} \cap L_i$ for $1 \le i \le d$ and an output $x_0 \in L_0 \cap L_d$. Let

$$\mathcal{M}(x_0; x_d, \dots, x_1)$$

be the set of orientation-preserving immersed disks with $d+1$ convex corners, modulo orientation-preserving reparametrization. Their boundary arcs follow, in counterclockwise order,

$$x_0 \xrightarrow{L_0} x_1 \xrightarrow{L_1} x_2 \longrightarrow \cdots \longrightarrow x_d \xrightarrow{L_d} x_0.$$

Every boundary arc has a specified continuous lift to the normalization of the corresponding curve. In particular, it cannot switch branches at a self-intersection along the interior of an edge. The image of a polygon need not be embedded. Convexity means that its interior angle at each corner lies strictly between 0 and π.



$\mu^2(x_2, x_1)$

Tracing tangent lines around the boundary, using (1) at the corners, gives

$$|x_0| = |x_1| + \cdots + |x_d| + 2 - d. \tag{3}$$

This is the degree formula behind the A_∞ operations.

4.2 Signs

Let $\delta_i(u) \in \{0,1\}$ be 0 if the boundary direction on the L_i edge agrees with the grading-induced orientation of L_i, and 1 if it disagrees. Let $n_i(u)$ count passages through the marked point on that edge, with multiplicity. If the edge is on a line, put $n_i(u) = 0$. The sign exponent is

$$s(u) = \sum_{i=0}^{d} \sigma_i n_i(u) + \sum_{i=1}^{d} |x_i|\delta_i(u) + |x_0|\delta_d(u) \pmod 2. \tag{4}$$

Here σ_i is the spin label on the component traversed by the edge. Thus every odd input tests the orientation of the edge on its target curve, and the odd output tests the L_d edge. The output contribution must be included. If all marked points are active, this is Abouzaid's polygon sign rule.

For a bigon, $|x_0| = |x_1| + 1$, so the corner part of the sign is simply $\delta_1(u)$. In words: the bigon contributes a minus sign when its L_1 edge runs against the orientation of L_1, together with the marked-point signs.

4.3 Local-system weights and the operations

Let $a_i \in \mathrm{Hom}((E_{i-1})_{x_i}, (E_i)_{x_i})$ be the input at x_i, and let τ_i be parallel transport along the L_i edge in the boundary direction. The local-system contribution of u is the map

$$H_u(a_d, \ldots, a_1) = \tau_d \circ a_d \circ \tau_{d-1} \circ a_{d-1} \circ \cdots \circ \tau_1 \circ a_1 \circ \tau_0 \in \mathrm{Hom}((E_0)_{x_0}, (E_d)_{x_0}).$$

Set

$$\mu^d(a_d, \ldots, a_1) = \sum_{x_0} \sum_{u \in \mathcal{M}(x_0; x_d, \ldots, x_1)} (-1)^{s(u)} T^{\int u^*\omega} H_u(a_d, \ldots, a_1). \tag{5}$$

With trivial local systems, H_u is just the scalar product of the inputs, in the output summand. In the exact integral version omit $T^{\int u^*\omega}$. Equation (3) says that

$$\mu^d : CF^*(L_{d-1}, L_d) \otimes \cdots \otimes CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_d) \quad \text{has degree } 2-d.$$

In particular, μ^1 counts bigons and is the Floer differential; μ^2 counts triangles and is the product; μ^3 counts quadrilaterals and supplies the homotopy for associativity. In this unobstructed construction $\mu^0 = 0$.

4.4 Finiteness and area weights

For a fixed finite transverse tuple, the polygon sums have only finitely many terms of bounded area. One way to see the relevant compactness on a surface is to keep track of the lifted boundary arcs. On an $\mathbb{R}$ component, an immersed boundary arc is monotone in its normalization and lies between its prescribed endpoints. On a circle component its image stays in the compact image of that circle. Thus all boundaries lie in a fixed compact finite graph $K \subset \Sigma$.

The multiplicity of an immersed disk is nonnegative and locally constant on $\Sigma \setminus K$. It vanishes on non-relatively-compact complementary regions, so all such disks lie in a common compact subset. Each region that occurs has positive area; an area bound bounds its multiplicity. There are finitely many regions, and only finitely many ways of assembling the resulting finite sheets into a disk with the prescribed corners. This gives the required finiteness and makes (5) a Novikov-convergent sum. The valuation-zero condition on local-system holonomy preserves this convergence.

For exact curves with primitives f_i, put

$$A(x_i) = f_i(x_i) - f_{i-1}(x_i), \qquad A(x_0) = f_d(x_0) - f_0(x_0),$$

where each value is taken on the specified branch. Stokes' theorem gives

$$\int u^*\omega = A(x_0) - \sum_{i=1}^{d} A(x_i). \tag{6}$$

Hence for fixed corners all polygons have the same area, and the sum is finite. This justifies counting over $\mathbb{Z}$. After extending scalars to $\Lambda_{\mathbb{Z}}$, the change of basis $\widehat{x} = T^{A(x)}x$ removes the area weights from the weighted operations as well.

5 The A_∞ equations

For homogeneous inputs $a_i \in CF^*(L_{i-1}, L_i)$, the equations are

$$\sum_{s=1}^{d}\sum_{r=0}^{d-s}(-1)^{\sum_{i=1}^{r}(|a_i|-1)}\mu^{d-s+1}(a_d, \ldots, a_{r+s+1}, \mu^s(a_{r+s}, \ldots, a_{r+1}), a_r, \ldots, a_1) = 0. \tag{7}$$

All the curves occurring in this equation belong to a single transverse tuple, so every term is defined.

For $d = 1$, this says $(\mu^1)^2 = 0$, and we define

$$HF^*(L_0, L_1) = H^*(CF^*(L_0, L_1), \mu^1).$$

For $d = 2$, it says

$$\mu^1\mu^2(b, a) + \mu^2(b, \mu^1(a)) + (-1)^{|a|-1}\mu^2(\mu^1(b), a) = 0.$$

Thus the product descends to cohomology. With these conventions the ordinary cohomological composition is

$$[b] \circ [a] = (-1)^{|a|}[\mu^2(b, a)].$$

The $d = 3$ equation makes this composition associative; on cochains the failure of associativity is accounted for by μ^3 and the terms in which μ^1 hits an input.

The geometric proof is the same at every order. A term in (7) is a pair of convex polygons glued along an intermediate corner. The relevant one-dimensional configurations are represented combinatorially by disks with a nonconvex corner; their two resolutions give the paired terms. Areas add under gluing, and parallel transport composes, so the two terms have the same area and local-system weight. The sign rule (4), together with the displayed A_∞ sign, gives opposite signs to the two decompositions. They therefore cancel. The excluded contractible circles and fishtails rule out the unwanted disk degenerations. This is the surface version of the boundary-of-a-one-dimensional-moduli-space argument.

Exercise 1 ($d^2 = 0$). Let $d = \mu^1$. Prove in detail that $d^2 = 0$, including pictures of the possible configurations of two composable bigons and their paired decompositions. Explain how the contributions cancel and where the unobstructedness assumptions are used. You may ignore signs by working in characteristic 2, and you may take all local systems to be trivial.

For all exercises: You are always welcome to consult the literature, but the write-up must be your own.

6 Why a pre-A_∞ category?

The operations above are defined only for transverse tuples. In particular, (L, L) is not a transverse pair, so this definition does not give a complex $CF^*(L, L)$ or a literal identity morphism. This is why we first obtain a *pre-A_∞ category.*

More explicitly, its data consist of objects, collections of transverse tuples closed under taking subsequences, morphism complexes for transverse pairs, and the operations μ^d for transverse $(d+1)$-tuples, satisfying (7). The objects here are the unobstructed graded curves with the spin and local-system data described above.

6.1 Quasi-isomorphisms and the extension property

In the compact-curve case, the construction has the additional unital extension property of Abouzaid's pre-category. A closed degree-zero morphism $e \in CF^0(L, L')$ is a quasi-isomorphism if composing with its cohomology class induces isomorphisms on all Floer cohomology groups for which the relevant triples are transverse. These maps are

$$HF^*(L', B) \longrightarrow HF^*(L, B), \qquad HF^*(B, L) \longrightarrow HF^*(B, L').$$

Equivalently, the appropriately signed left and right composition maps are quasi-isomorphisms of complexes.

Given an object L and finitely many transverse tuples S_ν, one can find perturbations L^-, L^+ and degree-zero quasi-isomorphisms

$$L^- \longrightarrow L \longrightarrow L^+$$

such that every tuple (L^-, S_ν, L^+) is transverse. The perturbations are small Hamiltonian perturbations, with the brane data transported along them. Locally the intersections near the diagonal are described by a Morse function on the normalization, and the identity section in $H^0(\widetilde{L}; \operatorname{End} E)$ supplies the continuation element. The small-triangle calculation below, together with Hamiltonian invariance, shows that composition with this element induces the continuation isomorphisms. This is the replacement for literal units in the pre-category formalism.

It allows us to replace curves by transverse representatives whenever needed. One can then pass to an ordinary cohomologically unital A_∞ model, form twisted complexes, and split idempotents. This is the meaning of $D^\pi \operatorname{Fuk}(\Sigma)$ in this setting.

The small-triangle calculation. Here is the calculation just invoked. First consider an embedded circle L. In coordinates (s, r) with $\omega = ds \wedge dr$, take

$$L = \{r = 0\}, \qquad L' = \{r = -\epsilon f'(s)\}, \qquad 0 < \epsilon \ll 1,$$

where s is periodic and f is Morse. This is a Hamiltonian perturbation. The minima p_i of f have degree zero in $CF(L, L')$, and its maxima q_i have degree one, by (2). Arrange that all minima have value 0 and all maxima have value 1. Each thin bigon from an adjacent minimum to a maximum has area ϵ. Transporting the brane data from L to L', set

$$e = \sum_i \mathrm{id}_{E_{p_i}} p_i.$$

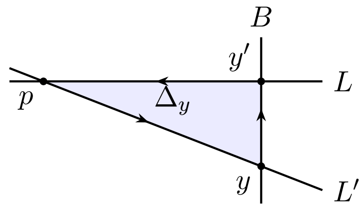
At each maximum the two incident bigons have opposite signs and the same parallel-transport map, so their contributions to $\mu^1(e)$ cancel. Thus e is closed and represents the identity section.

With arbitrary critical values the corresponding formula is $e = \sum_i T^{\epsilon f(p_i)} \mathrm{id}_{E_{p_i}} p_i$: both terms at q_i then have weight $T^{\epsilon f(q_i)}$. In the exact integral version all these area weights are omitted.

Now let B be transverse to L. A local model for one of the triangles is

$$L = \{r = 0\}, \qquad L' = \{r = -\epsilon s\}, \qquad B = \{s = b\}, \qquad b > 0.$$

Its vertices are $p = (0,0)$, $y = (b, -\epsilon b) \in L' \cap B$, and $y' = (b, 0) \in L \cap B$. The counterclockwise boundary runs $y' \to p \to y \to y'$, on L, L', B respectively.



There is one immersed triangle with this image, up to reparametrization, and its area is $\epsilon b^2/2$. Put the spin marked points outside the picture and use transported local-system frames. Since $|p| = 0$ and $|y| = |y'|$, the two sign terms on the B edge cancel modulo 2. Its contribution is therefore

$$\mu^2(y,p)\big|_{\Delta_y} = T^{\epsilon b^2/2} y'.$$

Over $\mathbb{Z}$ in the exact convention the coefficient is simply $+1$. This is the meaning of the small-triangle calculation: composition with the local degree-zero element identifies nearby intersections.

For a fixed finite family of test curves, choose the oscillations of f and the size of the perturbation so that these triangles occur at every test intersection. The resulting composition maps are diagonal chain isomorphisms, with invertible signed area weights; this is the adapted perturbation argument of [1, Lemma 4.4 and Corollary 4.6]. For an immersed curve the construction is made on its lifted branches, with the boundary lifts retained. For general Hamiltonian perturbations, subdivision and the polygon homotopies identify the products with continuation on cohomology. In our sign conventions the two maps are

$$y \longmapsto \mu^2(y, e), \qquad x \longmapsto (-1)^{|x|} \mu^2(e, x).$$

Quadrilaterals provide the homotopies for composing successive steps. Thus the reference to small polygons also includes these compatibility homotopies; it does not assert that every polygon for arbitrary perturbations is small.

6.2 The noncompact curves and the next step

For proper noncompact curves with finite intersections, the literal polygon construction and its A_∞ relations above still make sense. However, properness alone does not specify the behavior of perturbations at infinity, and we do not assert the unital extension property for the unrestricted collection of all such curves. To obtain a unital category including them, we choose the appropriate conditions at infinity and the corresponding perturbation or wrapping convention.

For the next part of the lecture we will take $\Sigma = T^*S^1$, use its exact symplectic structure, and allow exact curves conical at infinity. Wrapping provides the additional morphisms at the ends. The local picture of bigons, polygons, gradings, and signs developed here will remain the basis for the calculation of $\mathcal{W}(T^*S^1)$ and its HMS interpretation. The compact two-torus example instead retains the Novikov area weights.

7 The wrapped category of T^*S^1

7.1 The exact cylinder and its objects

Write

$$T^*S^1 = \mathbb{R}_p \times (\mathbb{R}/\mathbb{Z})_q, \qquad \theta = p\,dq, \qquad \omega = d\theta = dp \wedge dq.$$

Thus we use the *standard tautological exact structure.* Its Liouville vector field is $Z = p\partial_p$, and its two ends are $p \to +\infty$ and $p \to -\infty$. We choose the standard oriented line field $\eta = \mathbb{R}\partial_p$, oriented by ∂_p.

This is a model for the symplectic geometry of $\mathbb{C}^*$. More explicitly, under $z = e^{p+2\pi i q}$ the tautological form becomes

$$\theta = \frac{\log|z|}{2\pi}\, d\arg z, \qquad \omega = \frac{1}{2\pi}\, d\log|z| \wedge d\arg z.$$

This is the complete exact cylindrical structure on $\mathbb{C}^*$, with its unit circle as the zero section. It is a representative of the usual finite-type Stein geometry of the affine variety $\mathbb{C}^*$.

The objects of $\mathcal{W}(T^*S^1)$ are exact graded curves, compact or conical at infinity, with the spin and local-system data used above. We continue to allow unobstructed immersions: each component lifts to an embedded line in the universal cover. Conical means that, outside a compact set, the curve is a finite union of rays

$$\{q = q_j,\ p \geq R\} \quad \text{or} \quad \{q = q_j,\ p \leq -R\}.$$

The primitive of $\iota^*\theta$ is constant on each such ray; these constants need not all agree. We work over $\mathbb{Z}$, with rank-one local systems as before. Two basic objects are the zero section $S = \{p = 0\}$ and a cotangent fiber $F = \{q = 0\}$.

The wrapped Fukaya category is a $\mathbb{Z}$-graded A_∞ category. Here we describe only the cohomology of its morphism complexes and their cohomological composition:

$$H^* \operatorname{Hom}_{\mathcal{W}}(L_0, L_1) = HW^*(L_0, L_1).$$

The integer grading and exactness are important: no Novikov completion is needed in this section.

7.2 Positive wrapping and ordinary direct limits

Choose a smooth even convex function $h : \mathbb{R} \to \mathbb{R}$ with $h'(p) = 1$ for $p \geq 1$, $h'(p) = -1$ for $p \leq -1$, and $h''(p) > 0$ for $-1 < p < 1$. Put $H = h(p)$. With the convention $\iota_{X_H}\omega = -dH$, its flow is

$$\phi^w(p, q) = (p, q + wh'(p)), \qquad w \geq 0.$$

Thus positive wrapping increases q at the positive end and decreases q at the negative end. Write $L^w = \phi^w(L)$, with its transported brane data.

For a generic value of w, the ends of L_0^w and L_1 are disjoint. After a small perturbation in a compact set, they form a transverse pair with finitely many intersections. Hence the preceding surface construction already defines

$$V_w(L_0, L_1) := HF^*(L_0^w, L_1).$$

We choose each finite collection of wrapping times generically, so that every pair of ends used below is disjoint; arbitrarily small changes of the times are allowed when necessary. We then make the finite tuples transverse by small compactly supported perturbations before using their polygon operations.

For $w < w'$, positive wrapping gives a degree-zero continuation class

$$c_{w',w} \in HF^0(L_0^{w'}, L_0^w).$$

For a sufficiently small positive step, this is the local unit class near the diagonal: in the Morse picture it is the class of the identity section in $H^0(\widetilde{L}_0; \operatorname{End} E_0)$. Longer steps are obtained by subdividing the isotopy and composing these classes. The transition map is

$$\kappa_{w',w} : V_w(L_0, L_1) \longrightarrow V_{w'}(L_0, L_1), \qquad x \longmapsto x \circ c_{w',w}. \tag{8}$$

In our sign conventions this is $[\mu^2(x, c_{w',w})]$. The continuation classes satisfy

$$c_{w'',w} = c_{w',w} \circ c_{w'',w'}, \qquad \kappa_{w'',w} = \kappa_{w'',w'}\kappa_{w',w}$$

on cohomology. They are not asserted to be quasi-isomorphisms before passing to the wrapped category.

We define, in this particular cylinder setting,

$$HW^*(L_0, L_1) := \varinjlim_{w\to+\infty} HF^*(L_0^w, L_1). \tag{9}$$

This is an *ordinary direct limit of graded abelian groups*, using (8). One may take any cofinal increasing sequence of generic wrapping times. An element is represented at a finite stage, and two representatives agree if their images agree at a sufficiently large common stage.

This is the Lagrangian version of the direct-limit viewpoint in Seidel's *A biased view of symplectic cohomology* [3, §3e]. Seidel formulates that viewpoint for symplectic cohomology. For the continuation-class formulation used here, see also [5, §§3.3–3.4].

7.3 The cohomological product

Suppose that

$$x \in HF^*(L_0^a, L_1), \qquad y \in HF^*(L_1^b, L_2).$$

Apply ϕ^b to a representative of x. This identifies it with

$$(\phi^b)_* x \in HF^*(L_0^{a+b}, L_1^b).$$

Now use the triangle product of the transverse triple (L_0^{a+b}, L_1^b, L_2). The wrapped composition is represented by

$$y \circ x = (-1)^{|x|}[\mu^2(y, (\phi^b)_* x)] \in HF^*(L_0^{a+b}, L_2). \tag{10}$$

If necessary, move to slightly larger generic stages before applying this formula. Naturality of continuation and the A_∞ relations show that the result is independent of the representatives and is associative on cohomology. The small positive continuation classes give the identity morphisms in the limit.

Thus *all the maps needed for this cohomological description are already present in the surface pre-A_∞ category*: bigon differentials, triangle products, composition with continuation elements, and the polygon homotopies expressing their compatibility. We do not construct the chain-level wrapped A_∞ structure by taking an ordinary limit of complexes. Our use of ordinary limits is only the cohomological definition (9) and the product (10).

7.4 Classification of the geometric objects

We first state precisely the geometric equivalence relation. An *exact isotopy of immersed curves* here means a regular isotopy through exact immersions with embedded universal-cover lifts; isolated self-tangencies or triple points are allowed during the isotopy. For noncompact curves we hold the end rays fixed, but do not fix the constants of a primitive on those rays.

Temporarily orient each connected component. Its classification is:

Compact component	A nonzero winding number $n \in \mathbb{Z}$.
Line joining opposite ends	The two ordered end rays and a relative winding number $k \in \mathbb{Z}$.
Line returning to the same end	The two ordered end rays and a relative winding number $k \in \mathbb{Z}$.

To define k, choose real lifts of the angular coordinates of the two end rays. Lift the curve starting at the chosen first lift. Its other end has angular coordinate equal to the chosen second lift plus k. Changing the reference lifts translates the integer label. For two ends on the same side, the two end rays must be distinct, as required by the transverse-self-intersection condition.

For the compact case, lift a winding-n curve to the cyclic $|n|$-fold cover of the cylinder. It is an embedded essential circle there. Its action is zero by exactness, so the elementary area classification of essential circles gives an exact isotopy to a small generic perturbation of the zero section in that cover. Projecting back gives a generic immersed representative of the n-fold zero section. For example, for $n > 0$ one can use

$$t \longmapsto (\varepsilon g(t), nt \bmod \mathbb{Z}), \qquad t \in \mathbb{R}/\mathbb{Z}, \quad \int_0^1 g(t)\,dt = 0,$$

with g generic. Its action is $n\varepsilon \int_0^1 g(t)\,dt = 0$.

For a line component, the lift is an embedded proper arc in the universal strip with the prescribed lifted ends. Such arcs are isotopic relative to their ends. Projecting gives the claimed regular isotopy, and every intermediate immersion is exact because $H^1(\mathbb{R};\mathbb{R}) = 0$. This also explains why returning arcs must be included. Finite unions are described component by component. In particular, the compact immersed objects include all nonzero winding numbers; embedded compact curves have only $|n| = 1$.

To recover the branes, add the grading, spin structure, and local system. For a fixed induced orientation, compatible gradings differ by even integers. Equivalently, choose an unoriented reference representative and allow arbitrary integer shifts; an odd shift reverses its induced orientation. A rank-one integral local system on a circle has holonomy ± 1, and on a line it is trivializable.

This classification uses the stated exact isotopy of immersions. Ambient Hamiltonian isotopy is finer, since it preserves self-intersection data. If primitive constants are also fixed at infinity, the difference of their values, or equivalently $\int_{\mathbb{R}} \iota^*\theta$, is an additional invariant for a line component. Quasi-isomorphism in the wrapped category is a separate equivalence relation; it is not being classified here.

7.5 The endomorphism algebra of a cotangent fiber

Take $F = \{q = 0\}$ with tangent angle zero relative to $\eta = \mathbb{R}\partial_p$ and with its unique spin structure and trivial rank-one local system. For $w > 0$ not an integer,

$$F^w = \{q = wh'(p) \bmod \mathbb{Z}\}.$$

Its intersections with F are labeled by integers m with $|m| < w$:

$$x_m^w : \qquad h'(p(x_m^w)) = m/w.$$

In the oriented coordinates (p, q) the tangent of F^w is $(1, wh''(p))$, while the tangent of F is $(1, 0)$. The grading formula (2) therefore puts every x_m^w in degree zero. Consequently

$$HF^*(F^w, F) = \bigoplus_{|m|<w} \mathbb{Z}x_m^w, \qquad HF^j(F^w, F) = 0 \quad (j \neq 0). \tag{11}$$

There are no bigons between distinct generators, as is also clear by lifting to the universal cover.

For example, take the cofinal sequence $w_N = N\alpha$, where $0 < \alpha < 1$ is irrational. Here is the continuation calculation explicitly. Write $w' = w + \alpha$. The pair $(F^{w'}, F^w)$ has just one intersection: its label is 0, since $\alpha h'(p)$ can be an integer only when $h'(p) = 0$. It gives the degree-zero continuation class $c_{w',w}$.

For $|m| < w$, put $s = h'(p)$ and use the three lifts

$$F : \ q = 0, \qquad F^w : \ q = ws - m, \qquad F^{w'} : \ q = w's - m.$$

The relevant lifted corners are

$$c = (0, -m), \qquad x_m^w = (m/w, 0), \qquad x_m^{w'} = (m/w', 0).$$

For $m \neq 0$ these bound one positively oriented triangle, with boundary order $x_m^{w'} \to c \to x_m^w \to x_m^{w'}$. Since $h'' > 0$ in this region, the change from p to s preserves orientation. All degrees are zero, there are no spin marked points, and we are using exact integral coefficients. The triangle therefore has coefficient $+1$. The chosen lifts force the output label to remain m; there are no other outputs.

When $m = 0$, remove the coincident corners by a small generic perturbation. For example, replace the $F^{w'}$ lift by $q = w's - m + \delta$, where $0 < \delta \ll \alpha$. The three corners then have s-coordinates $-\delta/\alpha$, 0, and $-\delta/w'$ when $m = 0$, and again bound a single positive triangle. Thus, for every old generator,

$$\mu^2(x_m^w, c_{w',w}) = x_m^{w'}, \qquad |m| < w.$$

This is the small-triangle argument in this particular wrapped model. Consequently

$$HW^0(F, F) = \bigoplus_{m\in\mathbb{Z}} \mathbb{Z}x_m, \qquad HW^j(F, F) = 0 \quad (j \neq 0).$$

This is a direct sum: its elements are finite linear combinations.

The product is just as explicit. For inputs with labels m and n at times a and b, respectively, work upstairs and put $s = h'(p)$. The three relevant lifted curves are

$$q = 0, \qquad q = bs - n, \qquad q = (a + b)s - (m + n).$$

Their input vertices have s-coordinates m/a and n/b, and the output vertex has

$$s = \frac{m+n}{a+b} = \frac{a}{a+b}\frac{m}{a} + \frac{b}{a+b}\frac{n}{b}.$$

These lines bound the unique contributing triangle. All corner degrees are zero and there are no spin marked points, so its sign is $+1$. Small generic perturbations handle coincident vertices, including the unit calculation. Therefore

$$x_n \circ x_m = x_{m+n}, \qquad 1 = x_0,$$

and, on setting $z = x_1$, we obtain the cohomological algebra

$$HW^*(F,F) \cong \mathbb{Z}[z, z^{-1}], \qquad |z| = 0. \tag{12}$$

Finally, the endomorphism A_∞ algebra is *formal*:

$$\mathrm{Hom}_{\mathcal{W}}(F,F) \simeq \mathbb{Z}[z, z^{-1}]$$

as an A_∞ algebra, with the usual multiplication on the right and all other operations zero. Indeed, its cohomology is free over $\mathbb{Z}$, so we can transfer the A_∞ structure to a minimal model on cohomology. Since the cohomology is concentrated in degree zero and μ^d has degree $2-d$, every operation except μ^2 vanishes for degree reasons. The product is the Laurent-polynomial multiplication computed above.

For more details and pictures of this computation, see Auroux's *A beginner's introduction to Fukaya categories* [4, §§4.1–4.2, especially Figure 12 and equation (4.4)].

8 Mirrors in dimension one

8.1 The cylinder and marked points

The first mirror pair is

$$X = \mathbb{C}^* \quad \longleftrightarrow \quad \mathbb{G}_{m,\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[z, z^{-1}].$$

We have already encountered the coordinate ring on the right as the endomorphism algebra of a cotangent fiber.

Adding a marked point gives the relative version of the picture:

$$(X, D) = (\mathbb{C}^*, \{1\}) \quad \longleftrightarrow \quad \operatorname{Spec} \frac{\mathbb{Z}[x,y,Q]}{(xy-Q)} \longrightarrow \operatorname{Spec} \mathbb{Z}[Q].$$

The parameter Q records intersections of polygons with the marked point. The fiber at $Q = 0$ is the union of the two coordinate axes; when Q is invertible the fiber is a multiplicative group.

For $D = \{p_1, \ldots, p_n\} \subset \mathbb{C}^*$, give each point its own parameter Q_i, and put $B = \mathbb{Z}[Q_1, \ldots, Q_n]$. Glue the charts

$$U_i = \operatorname{Spec} \frac{B[x_i, y_i]}{(x_i y_i - Q_i)}, \qquad 1 \le i \le n,$$

by identifying the open set $y_i \neq 0$ in U_i with $x_{i+1} \neq 0$ in U_{i+1}, using

$$x_{i+1} y_i = 1.$$

Thus the overlap is

$$V_i = \operatorname{Spec} \frac{B[y_i, x_{i+1}]}{(y_i x_{i+1} - 1)}.$$

The remaining coordinates on it are $x_i = Q_i x_{i+1}$ and $y_{i+1} = Q_{i+1} y_i$. The gluing diagram is

$$U_1 \longleftarrow V_1 \longrightarrow U_2 \longleftarrow V_2 \longrightarrow \cdots \longleftarrow V_{n-1} \longrightarrow U_n.$$

After setting all $Q_i = 0$ and base changing to $\mathbb{C}$, the central fiber is a chain with n nodes. It has $n-1$ copies of $\mathbb{P}^1_{\mathbb{C}}$ and two affine-line end components.

8.2 From the infinite chain to the Tate curve

We can instead take charts indexed by $i \in \mathbb{Z}$, put $Q_i = q$, and glue in the same way. The central fiber is then an infinite chain of projective lines, with a translation action of $\mathbb{Z}$.

To form the desired quotient, first complete along $q = 0$. The translation quotient of this formal scheme is a proper formal curve over $\operatorname{Spf} \mathbb{Z}[[q]]$. An ample line bundle allows us to algebraize it by Grothendieck existence. The result is the *Tate curve*

$$\mathcal{T}_1 \longrightarrow \operatorname{Spec} \mathbb{Z}[[q]].$$

Its central fiber is a nodal rational curve, and it is a smooth elliptic curve after inverting q. The formal completion before taking the quotient is essential in this construction. See [8, §2] for the periodic gluing construction and its algebraization.

Proposition 8.1 (Tate uniformization)**.** *Let K be a complete nonarchimedean valued field. An elliptic curve E/K with split multiplicative reduction is obtained from $\mathcal{T}_1$ by a unique parameter $\tau \in K$ with $0 < |\tau| < 1$, through the map*

$$\mathbb{Z}[[q]] \longrightarrow K, \qquad q \longmapsto \tau.$$

Analytically, it has the uniformization

$$E^{\mathrm{an}} \cong \mathbb{G}_{m,K}^{\mathrm{an}} / \tau^{\mathbb{Z}}.$$

In particular, the familiar notation $K^*/\tau^{\mathbb{Z}}$ describes its K-points. See Tate [9] and [10, §19].

8.3 The symplectic two-torus

Let T^2_A be the symplectic two-torus of total area $\int_{T^2} \omega = A > 0$. Its mirror is

$$E_A := \mathcal{T}_1 \times_{\operatorname{Spec} \mathbb{Z}[[q]]} \operatorname{Spec} \Lambda_{\mathbb{Q}}, \qquad q \longmapsto T^A.$$

Thus

$$T^2_A \quad \longleftrightarrow \quad E_A.$$

The corresponding relative pictures are

$$(T^2, \{p\}) \quad \longleftrightarrow \quad \mathcal{T}_1, \qquad (T^2, \{p_1, \ldots, p_n\}) \quad \longleftrightarrow \quad \mathcal{T}_n / \mathbb{Z}[[Q_1, \ldots, Q_n]],$$

where again each marked point has a separate weight. Here $\mathcal{T}_n$ is the *n-Tate curve.*

Optional exercise (the n-Tate curve). Describe $\mathcal{T}_n$ using the infinite chain with periodic parameters $Q_{i+n} = Q_i$ and the translation by n charts. Identify its central fiber and the Tate parameter of its smooth fibers. You may use the literature, for example [8, §2], but write the explanation in your own words.

9 Homological mirror symmetry

For the sake of time, we omit the versions enhanced by large volume limit data (see Lekili–Perutz [7, Theorem A and §1.1]; the version with several marked points is due to Lekili–Polishchuk [8]). The two statements are

$$D^\pi \mathcal{W}(\mathbb{C}^*) \simeq D^b \operatorname{Coh}(\mathbb{G}_{m,\mathbb{Z}}), \tag{13}$$

$$D^\pi \operatorname{Fuk}(T^2_A) \simeq D^b \operatorname{Coh}(E_A). \tag{14}$$

The first equivalence is over $\mathbb{Z}$; the second is over $\Lambda_{\mathbb{Q}}$. As in the overview, these statements have A_∞ or dg enhancements.

9.1 Restricted Yoneda and the cotangent fiber

Here is the way to understand (13). Let $R \subset T^*S^1$ be a cotangent fiber, and set

$$\mathcal{A} = \operatorname{Hom}_{\mathcal{W}(\mathbb{C}^*)}(R, R).$$

There is a restricted Yoneda functor

$$\mathcal{Y}_R : \mathcal{W}(\mathbb{C}^*) \longrightarrow \operatorname{Mod}_{A_\infty}(\mathcal{A}), \qquad L \longmapsto \operatorname{Hom}_{\mathcal{W}(\mathbb{C}^*)}(R, L).$$

We use right modules: their structure maps come from precomposition with endomorphisms of R, including the higher compositions.

The nontrivial input is that R *split-generates* $\mathcal{W}(\mathbb{C}^*)$. Equivalently, $\mathcal{Y}_R$ is cohomologically fully faithful and takes values in perfect modules:

$$\mathcal{W}(\mathbb{C}^*) \hookrightarrow \operatorname{Perf}_{A_\infty}(\mathcal{A}).$$

Here a perfect module is a direct summand, up to quasi-isomorphism, of a finite twisted complex of free modules. The cotangent-fiber generation theorem is due to Abouzaid [6, Theorem 1.1]. For the immersed surface model used here, the generation argument uses boundary lifts to the normalizations; the no-fishtail condition excludes the additional disk bubbling.

9.2 Passing to perfect modules

Write $\operatorname{Perf}_{A_\infty}(\mathcal{C})$ for the category obtained from the Yoneda image of an A_∞ category $\mathcal{C}$ by taking finite twisted complexes and splitting idempotents. Its homotopy category is $D^\pi\mathcal{C}$.

The restricted Yoneda functor extends to a cohomologically fully faithful functor

$$\operatorname{Perf}_{A_\infty}(\mathcal{W}(\mathbb{C}^*)) \longrightarrow \operatorname{Perf}_{A_\infty}(\mathcal{A}).$$

Its image contains the free rank-one module $\mathcal{Y}_R(R) = \mathcal{A}$. Since this module split-generates the target, the extended functor is a quasi-equivalence. The fiber calculation and formality from the preceding section now give

$$\begin{aligned} \operatorname{Perf}_{A_\infty}(\mathcal{W}(\mathbb{C}^*)) &\simeq \operatorname{Perf}_{A_\infty}(\mathcal{A}) \\ &\simeq \operatorname{Perf}_{A_\infty}(\mathbb{Z}[z, z^{-1}]) \\ &\simeq \operatorname{Perf}(\mathbb{G}_{m,\mathbb{Z}}). \end{aligned}$$

Finally, $\mathbb{Z}[z, z^{-1}]$ is a regular Noetherian ring of finite global dimension. Thus every bounded complex of coherent sheaves on $\mathbb{G}_{m,\mathbb{Z}}$ is perfect. Passing to homotopy categories gives (13).

In this construction the reference fiber R goes to the free module, and hence to the structure sheaf $\mathcal{O}_{\mathbb{G}_{m,\mathbb{Z}}}$. This is the sense in which our reference Lagrangian is mirror to the structure sheaf *by construction.*

Exercise 2 (images of geometric objects). What are the images of the objects of $\mathcal{W}(\mathbb{C}^*)$ under (13)? Use the geometric objects described in Section 7.4, including compact and noncompact curves, and keep track of their gradings and local systems. As always, you may consult the literature, but the writeup must be your own.

Exercise 3 (triangles and a theta series). Compute the following Floer triangle product, including pictures. Let

$$(T^2,\omega) = (\mathbb{R}^2/\mathbb{Z}^2, A\,dp\wedge dq), \qquad F=\{p=0\}, \quad R=\{q=0\}, \quad L=\{p+q=a \pmod{\mathbb{Z}}\},$$

where $A>0$ and $0<a<1$. Write

$$x\in CF(F,R), \qquad y\in CF(R,L), \qquad z\in CF(F,L)$$

for the unique intersection generators. Relative to the horizontal line field, take lifted tangent angles $1/2, 0, -1/4$ (in units of π) for F, R, L, respectively, so that all three generators have degree zero.

(i) Draw the circles and their lifts. Show that the triangles contributing to $\mu^2(y,x)$ are indexed by $n\in\mathbb{Z}$, with lifted vertices

$$(0,0), \qquad (n+a,0), \qquad (0,n+a).$$

Explain why both signs of $n+a$ contribute, even when a triangle overlaps itself after projection to the torus. Compute the areas.

(ii) Take trivial local systems and the nonbounding spin structure on each circle, denoting this configuration by s_0. Check the signs and obtain

$$\mu^2(y,x) = \Theta_{s_0}(a)z, \qquad \Theta_{s_0}(a) = \sum_{n\in\mathbb{Z}} T^{\frac{A}{2}(n+a)^2}.$$

Explain Novikov convergence. You may first do the unsigned count.

(iii) Change only the spin structure on F. Use the marked-point rule to obtain, up to one overall sign,

$$\Theta_{s_1}(a) = \sum_{n\in\mathbb{Z}} (-1)^n T^{\frac{A}{2}(n+a)^2}.$$

Explain why keeping s_0 and using holonomy -1 on F has the same effect.

(iv) Prove that $\Theta_{s_1}(1/2)=0$, and identify the cancelling pairs of triangles in your pictures.

You may consult the literature, but the write-up and pictures must be your own.

In the next lectures, we will outline how to understand the compact HMS statement (14) in a similar way, using Fukaya categories with support.

References

[1] M. Abouzaid, *On the Fukaya categories of higher genus surfaces*, Advances in Mathematics **217** (2008), 1192–1235. arXiv:math/0606598. See especially Sections 2–4 and Appendix C.

[2] V. de Silva, J. W. Robbin, and D. A. Salamon, *Combinatorial Floer homology*, Memoirs of the American Mathematical Society **230** (2014), no. 1080. arXiv:1205.0533.

[3] P. Seidel, *A biased view of symplectic cohomology*, Current Developments in Mathematics **2006** (2008), 211–253. arXiv:0704.2055.

[4] D. Auroux, *A beginner's introduction to Fukaya categories*, in *Contact and Symplectic Topology*, Bolyai Society Mathematical Studies **26** (2014), 85–136. arXiv:1301.7056.

[5] S. Ganatra, J. Pardon, and V. Shende, *Covariantly functorial wrapped Floer theory on Liouville sectors*, Publications Mathématiques de l'IHÉS **131** (2020), 73–200. doi:10.1007/s10240-019-00112-x.

[6] M. Abouzaid, *A cotangent fibre generates the Fukaya category*, Advances in Mathematics **228** (2011), 894–939. arXiv:1003.4449.

[7] Y. Lekili and T. Perutz, *Arithmetic mirror symmetry for the 2-torus*. arXiv:1211.4632.

[8] Y. Lekili and A. Polishchuk, *Arithmetic mirror symmetry for genus 1 curves with n marked points*, Selecta Mathematica **23** (2017), 1851–1907. arXiv:1601.06141.

[9] J. Tate, *A review of non-Archimedean elliptic functions* (1995). Available online.

[10] B. Conrad, *Modular curves*, Math 248B lecture notes, §19. Available online.