

The Concept of Manifold, 1850–1950

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1. Origin of the manifold concept

1.1. n -dimensional systems geometrized

In the early 19th century we find diverse steps towards a generalization of geometric language to higher dimensions. But they were still of a tentative and often merely metaphorical character. The analytical description of dynamical systems in classical mechanics was a field in which, from hindsight, one would expect a drive towards and a growing awareness of the usefulness of higher dimensional geometrical language.¹ But the sources do not, with some minor exceptions, imply such expectations. Although already Lagrange had used the possibility to consider time as a kind of fourth dimension in addition to the three spatial coordinates of a point in his *Mécanique analytique* (1788) and applied a contact argument to function systems in 5 variables by transfer from the 3-dimensional geometrical case in his *Theorie des fonctions analytiques* (1797, Section 3.5.25), these early indications were not immediately followed by others.

Not before the 1830-s and 1840-s do we find broader attempts to generalize geometrical language and geometrical ideas to higher dimensions: Jacobi (1834), e.g., calculated the volume of n -dimensional spheres and used orthogonal substitutions to diagonalize quadratic forms in n variables, but preferred to avoid explicit geometrical language in his investigations. Cayley's *Chapters in the analytical geometry of n dimensions* (1843) did use such explicit geometrical language – but still only in the title, *not* in the text of the article. It was the following decade about the middle of the century which brought the change. In a short time interval we find a group of authors using and exploring conceptual generalizations of geometrical thought to higher dimensions, without in general knowing about each other. Among them was Grassmann with his *Lineale Ausdehnungslehre* (1844) containing an explicit program for a new conceptual foundation for geometry on n -dimensional (lin-

¹Such is suggested in some older historical literature, e.g., in F. Klein's *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*. This is discussed with reference to more recent investigations in Section 2.3 below.

ear) extensional quantities,² Plücker with his *System der Geometrie des Raumes* (1846) and 4-dimensional line geometry in classical 3-space, and, in a certain respect most elaborated among these attempts, Schläfli with his *Theorie der vielfachen Kontinuität* (1851/1901), which was published only posthumously.³

Also leading mathematicians like Cauchy and Gauss started to use geometrizing language in \mathbb{R}^n in publications (Cauchy, 1847) or lecture courses (Gauß, 1851/1917). Gauss, in his lecture courses, even used the vocabulary of $(n - k)$ -dimensional manifolds (*Manigfaltigkeiten*), but still restricted in his context to affine subspaces of the n -dimensional real space (Gauß, 1851/1917, pp. 477ff.). There is no reason to doubt that Riemann got at least some vague suggestion of how to generalize the basic conceptual frame for geometry along these lines from Gauss and developed it in a highly independent way.

1.2. Riemann's n -dimensional manifolds

When Riemann presented his ideas on a geometry in manifolds the first time to a scientific audience in his famous *Habilitationsvortrag* (Riemann, 1854), he was completely aware that he was working in a border region between mathematics, physics, and philosophy, not only in the sense of the pragmatic reason that his audience was mixed, but by the very nature of his exposition.⁴ There was no linguistic or symbolical frame inside mathematics, which he could refer to, even only to formulate a general concept of manifold. So he openly drew on the resources of contemporary idealist, dialectical philosophy, in his case oriented at J.F. Herbart, to generalize the classical concept of *extended magnitude/quantity* for geometry and to “construct” the latter as only one specification from a more general concept.⁵ Basic to such a construction was, so Riemann explained to his audience, the presupposition of any “general concept” which allows in a logical sense precise individual determinations. From the extensional point of view such a concept would form a *manifold* and the individual modes of determination were to be considered, as Riemann explicitly stated, as the *elements* or the *points* of the manifold with either “discrete” or “continuous” transition from one to the other. Thus Riemann sketched the draft for a conceptual starting point for what later was to become general set theory (discrete manifolds)⁶ and topology (continuous manifolds).

Such concepts would gain mathematical value only if a sufficiently rich structure of (real or complex valued) functions on the manifold is available. Then it should be possible to describe the specification of points by the values of n properly chosen functions in a locally unique way (local coordinate system). That a change of coordinates would lead to locally invertible differentiable real functions, was not made explicit by him, but was to be understood from the context by careful listeners or readers. The distinction between *local simplicity* of manifolds, because of the presupposition of local coordinate systems, and *globally involved behaviour* was indicated by Riemann, but not particularly emphasized during the talk, although in other publications and manuscripts it was.⁷

² Hamilton's *quaternions* used 4-dimensionality for purely algebraic reasons, keeping geometry restricted to the 3-dimensional subspace of purely imaginary quaternions.

³ See [Kolmogorov and Yushkevitch, 1996].

⁴ For a detailed and very readable exposition of the width of Riemann's interests see [Laugwitz, 1996].

⁵ See [Scholz, 1982a].

⁶ For the line from Riemann via Dedekind and Cantor to general set theory see [Ferreirós, 1993, 1996].

⁷ Compare the next two sections of this article.

Of the utmost importance was Riemann's discussion of different conceptual levels – we would say *structures* – which can be considered on a given manifold. During his talk he exemplified these by the distinction between analysis situs (combinatorial topology of differential manifolds) and differential geometry. In his works on complex function theory he moreover pursued concrete investigations of complex and birational structure in the complex one-dimensional case (Riemann, 1851, 1857).⁸ And there are points in the latter publications, where Riemann indicated that it might be useful to work with even more “general concepts” of a continuous character, which would transcend the limits of the specific postulates for continuous manifolds introduced or at least presupposed in his Habilitations lecture. Thus in his dissertation Riemann (1851, p. 36) had already talked about infinite dimensional (real) function spaces and continuously varying conditions for functions in them, given by equations, which indicated nonlinear subsets in the dual of functions spaces. Moreover, Riemann had even already used the language of “continuous manifold” in this context without further specification what should be understood by that term.⁹ That was a drastic generalization of Gauss's finite dimensional linear submanifolds of \mathbb{R}^n and even far more general than the manifold concept as developed by Riemann in 1854.

That was three years before his Habilitations lecture. Three years after the latter, in his work on abelian functions, Riemann indicated how the complex/birational structure on a closed orientable surface of given genus p can be characterized by $3p - 3$ independent complex parameters describing a normalized branching behaviour over the complex plane. He thus started to explore the moduli space of Riemann surfaces of genus p and was cautious enough, not to talk about them as manifolds, but left it with a local description at generic points (Riemann, 1857, p. 122).

Thus Riemann presented an outline of a visionary program of a family of geometrical theories, bound together by the manifold concept, diversified by different conceptual and technical levels like topology, differential geometry, complex geometry, algebraic geometry of manifolds, and overarching the whole range from questions deep inside conceptual (“pure”) mathematics to the cognition of physical space and the nature of the constitution and interaction of matter. Here is not the place to follow all these branches; we rather concentrate on the tools for a topological characterization of manifolds with some digressions into the broader context.

1.3. Riemann on the topology of surfaces . . .

Riemann used different approaches in his studies of surfaces. Already in his dissertation he dealt with the connectivity of compact bounded surfaces. His goal was to introduce complex analytic functions on (Riemann) surfaces over a bounded region of the complex plane. For simply connected surfaces he used his famous argumentation by the Dirichlet principle to determine real and imaginary parts of a complex function by the potential equation and boundary value conditions.¹⁰ Here he characterized simple connectedness of

⁸ Some more details in [Dieudonné, 1974, pp. 42ff.; Gray, 1986; Scholz, 1980, pp. 51ff.].

⁹ “These conditions, the totality of which form a continuous manifold and which can be expressed by equations between arbitrary functions . . . still have to be limited or supplemented by single conditions for arbitrary constants . . .” (Riemann, 1851, p. 36).

¹⁰ See, e.g., [Bottazzini, 1986, pp. 229f.].

a surface F by the condition that F falls apart by any cross cut leading from one point of the boundary ∂F to another.¹¹ For not simply connected surfaces he introduced a connectivity number by a cut and count procedure.

If F can be dissected by m cuts along double-point free curves between the boundary of F or new boundary components arising from earlier cuts into n simply connected pieces, then, so Riemann argued, the difference $n - m$ is independent of the cutting procedure and a topological invariant. In fact Riemann's counting procedure can be read as a characterization of the Euler number $\chi(F)$ of the surface with each cross-cut increasing the Euler number by 1 (adding 2 zero cells and 1 one-cell) and leaving n simply connected surfaces $\chi(F) + m = n$, thus $\chi(F) = n - m$. By a specific choice of the dissection it is possible to reach exactly one simply connected piece at the end of the process, $n = 1$, giving the lowest number of cross cuts necessary, $m_0 = 1 - \chi(F)$. In this case Riemann would call the surface $(m_0 + 1)$ -fold connected.

In his later work on abelian integrals and functions (1857) Riemann considered surfaces over the whole (compactified) complex plane and thus closed orientable surfaces. In order to apply his early counting method for the connectivity number he showed that "recurrent cuts (Rückkehrschnitte)" do not change the latter (adding 1 zero cell and 1 one-cell) thus allowing him to apply the old method also to this case. His interest was now directed towards a different type of question: the periods of abelian integrals of first (or higher) kind, i.e. the characterization of multivaluedness of integrals of a holomorphic (or meromorphic) differential form ω on a closed Riemann surface $F \rightarrow P_1\mathbb{C}$. Starting from a general 2-dimensional version of the Gauss–Stokes theorem and the Cauchy–Riemann equations for the coefficients of the holomorphic form ω , he realized that (in modernized notation) $d\omega = 0$ and therefore for any set of closed (oriented) curves c_i , $1 \leq i \leq k$, forming a complete boundary of a part F' of the surface,

$$\bigcup_{i=1, \dots, k} c_i = \partial F',$$

the evaluation of the integral will give zero:¹²

$$\int_{c_1, \dots, c_k} \omega = \int_{F'} d\omega = 0.$$

Therefore, so Riemann concluded, the multivaluedness of integrals of holomorphic 1-forms (abelian integrals of the first kind) depends only (and still to a high degree in the case of meromorphic 1-forms, the abelian integrals of second and third kind)¹³ on the topology of the surface. So it was reasonable to characterize the topology of closed (orientable) surfaces in this context by a method of boundary relations between systems of curves, which from the later point of view reads as a first step towards a homology theory of 2-dimensional manifolds.

¹¹ Riemann thus used a purely homological characterization of simple connectedness, in contrast to the modern post-Poincaréan view. Compare the contribution by R. Vanden Eynde, this volume.

¹² (Riemann, 1857, pp. 91ff.), compare also the contribution of R. Vanden Eynde, this volume.

¹³ $\int_c \omega$ is an *abelian integral of second kind* if ω is a meromorphic differential form only with poles of order $m > 2$ and *abelian integral of third kind* if ω is a meromorphic form with poles of order 1 but with sum of residues 0.

For the purely topological part of his investigation Riemann did not take into account the orientation of curves or surface parts, thus simplifying the calculations. He introduced an equivalence between systems of curves C and C' if both together form the complete boundary of part of the surface $C + C' = \partial F'$, as in this case C and C' “achieve the same with respect to forming complete boundaries” with other curves (Riemann, 1857, p. 124). In slightly modernized reading Riemann thus worked with a geometrical description of bordance homology of submanifolds in F modulo 2, or, in another translation, with simplicial homology, if F is simplicially decomposed by cuts along the curves c_i such that the latter represent 2-cycles of the decomposition. Indeed Riemann showed that there is a well-determined number n of homologically independent curves, independent of the choice of the specific realization of the curve system, and that in the case of his surfaces this number is even, $n = 2p$ (Riemann’s notation (Riemann, 1857, p. 136)).

Of course, Riemann did not keep to the modulo 2 reduction of homology when working with integrals of differential forms. Once a complete set of generators of the homology c_1, \dots, c_{2p} and corresponding periods $w_i = \int_{c_i} \omega$ ($1 \leq i \leq 2p$) of a differential form were determined, he worked with integral linear combinations of the periods and thus (at least implicitly) with unreduced integral combinations of cycles (Riemann, 1857, pp. 137ff.). So the modulo 2 reduction was for him nothing more than a method to simplify the calculation of the topological invariants and in fact a result of a context dependent abstraction from orientation.

1.4. ... and on the connectivity of higher dimensional manifolds

In the edition from Riemann’s Nachlass Weber edited three fragments about analysis situs (Riemann, 1876a) in which Riemann explored first thoughts on the topological characterization of higher dimensional manifolds. These fragments can be dated with great probability to the time of Riemann’s work on his Habilitationsschrift, thus about the years 1852/1853.¹⁴ Here Riemann described the introduction of higher connectivity numbers using a bordance homological approach similar to the one later published in his theory of abelian functions and discussed in the last section. He considered closed connected submanifolds U_i , $1 \leq i \leq m$, of dimension n in a manifold M of dimension k ,¹⁵ which “taken each once, neither individually nor jointly” form the complete boundary of an $(n + 1)$ -dimensional submanifold, which means, expressed in more recent terminology, they form a set of homologically independent n -cycles.¹⁶

Riemann explicitly defined homological equivalence of n -cycles A and B , using the terminology of “transmutability” of A into B .¹⁷ Riemann then argued with an exchange argument which algebraically expressed would be the Steinitz lemma and the change of generators in the homology vector-space (take into consideration that Riemann worked

¹⁴ For more details see [Scholz, 1982b].

¹⁵ Riemann used the terminology “innere zusammenhängende unbegrenzte n -Strecke” for the U_i , without further specification of the objects considered. From a recent mathematical perspective “ n -Strecke” should perhaps not be understood as submanifold, but as “subvariety” admitting certain controlled singularities like the topological varieties in (Kreck, 1998).

¹⁶ As in the last section suppose there is a simplicial decomposition of M , in which the U_i represent n -cycles.

¹⁷ “Ein n -Streck A heisst in ein anderes B veränderlich, wenn durch A und durch Stücke von B ein inneres $(n + 1)$ -Streck vollständig begrenzt werden kann.” (Riemann, 1876a, p. 479).

mod 2): If V_i ($1 \leq i \leq m$) is another set of n -submanifolds which fulfill the same boundary conditions as the U_i , each of which forms jointly with some of the U_i the complete boundary of an $(n + 1)$ -submanifold, then with respect to the formation of bounding relations the V_i can be substituted step by step for the U_i and in the end the V_i and the U_i ($1 \leq i \leq m$) can be considered equivalent in the context of forming boundary relations inside the manifold M .

Riemann thus introduced the maximal number m of (mod 2) homologically independent n -cycles, i.e. the n th Betti number mod 2, and called the manifold M $(m + 1)$ -fold connected in dimension n (ibid.). In particular, he called M simply connected if all connectivity numbers (Betti numbers) mod 2 of M are zero, thus deviating from the modern, post Poincaréan, terminology (or better the other way round).¹⁸ He started to investigate the decomposition of a k -dimensional manifold by dissection along lower dimensional submanifolds, and tried to generalize his decomposition method from 1851 for surfaces to higher dimension, although he did not fully elaborate a symbolism to characterize types of such decompositions or topological invariants. The fragments leave no doubt, however, that already at the time of his Habilitationsvortrag he had a rather clear conceptual construction of Betti numbers modulo 2 in mind, taking into account the level of elaboration of symbolical characterization of manifolds. Enrico Betti seems to have been the only mathematician to whom he talked about these concepts in sufficient detail to transmit the essentials of his ideas. At least Betti was the only one in Riemann's lifetime, who understood what the latter was heading for.¹⁹

2. Dissemination of manifold ideas

2.1. The problem of how to characterize manifolds

The reception and assimilation of Riemann's concept of manifold to the mathematics of the 19th century was slow and inhibited by severe conceptual problems. Of course it was difficult to understand what a manifold in general should be. The easiest way was to translate it as a "number manifold" in the 1870-s and later. At that time the former real quantities had been arithmetically reconstructed by Meray, Cantor, Dedekind, and Weierstrass, and it appeared as perfectly clear to talk about concretely given submanifolds of \mathbb{R}^m or of projective spaces $P_m\mathbb{R}$ or $P_m\mathbb{C}$. Such submanifolds were in the easiest approach defined by inequalities as m -dimensional (usually connected) subsets in the works of Beltrami (1868a, 1868b) Helmholtz (1868), and even of the young Klein during his investigations on non-Euclidean geometry and the Erlangen program (1871).

That was of course a reduction of Riemann's intention and suppressed the distinction between local simplicity and global complexity of manifolds. That global behaviour was an essential ingredient for Riemann's concept, was most clearly understood in the 1860-s and 1870-s in the special context of geometric function theory and the dissemination of knowledge about the topology of Riemann surfaces (Lüroth, Clebsch, Neumann, Clifford et al.) An additional aspect was the problem of compactification of geometrical objects "in the infinite", which in a discussion between Schläfli and Klein was realized, when they

¹⁸ Compare the contribution of R. Vanden Eynde and footnote 11.

¹⁹ For the relationship between Riemann and Betti see [Bottazzini, 1977].

debated the difference between one-point compactification of the plane like \mathbb{C} to $P_1\mathbb{C}$ and line compactification of \mathbb{R}^2 to $P_2\mathbb{R}$ and its topological consequences (Schläfli, 1872; Klein, 1873, 1874–1876).

Only after discussions with Clifford on space forms during the 1873 meeting of the British Association for the Advancement of Science, Klein modified his earlier restricted concept of “manifold” and introduced the distinction between *relative* properties of a “number manifold”, which depended on the embedding, and *absolute* ones which did not, orientability given as an example for the latter, without going into technical details of how to identify the “absolute” properties (Klein, 1874–1876).

A more general characterization of “number manifolds” was the consideration of zero sets by equations (and inequalities), which usually were supposed to be nonsingular without further specification. This approach was taken by Betti (1871) in his paper on the topology of higher dimensional manifolds and by Lipschitz in his investigations of higher dimensional state spaces of mechanical systems (Lipschitz, 1872).²⁰ In Betti’s case global complexity was, of course, part of his object of study. The local simplicity, however, remained unanalyzed before the proof of the implicit function theorem, including an explicit statement of the condition under which it holds, became generally known. The theorem and its proof was developed by U. Dini during his lecture courses in the late 1870-s and spread in analysis courses and monographs during the late 1880-s and early 1890-s.²¹

Finally, a first, still clumsy and vaguely described, combinatorial approach to a characterization of n -dimensional manifolds was used by Klein’s student W. Dyck in addition to the characterization as a “number manifold”. Although starting as Klein had done from a submanifold M of \mathbb{R}^n , Dyck gave a vague description of how to build M from an n -ball E_n by cutting and pasting along submanifolds of type E_k isomorphic to k -balls (von Dyck, 1888, 1890). This process was not uniquely described in Dyck’s symbolism and presupposed sufficient intuition to be applied to a manifold defined by other means. It still sufficed for Dyck’s purpose, as his procedure served only as an aid for the topological characterization of manifolds, not for their definition or construction.

2.2. The changing concept of geometry

During the 19th century the perception, structure and role of geometry was fundamentally transformed. Classically there existed but one, Euclidean geometry, and its unique role in the framework of knowledge at the turn from early modernity to “high” modernity was paradigmatically exemplified in Kant’s philosophy of space. The breakthrough in the studies of the foundations of geometry has been described by I. Tòth as the shift from the “anti-Euclidean” hypothesis to the non-Euclidean point of view;²² it was realized independently, as is well known, by Gauss, Lobachevsky, and J. Bolyai. Until the 1860-s this change of view was shared only by a small minority of mathematicians, and was moreover conceptually still rather fragile, as long as *only* the theoretical structure of non-Euclidean

²⁰ For Lipschitz compare [Lützen, 1995] and the Section 2.3 below.

²¹ Two important publications for the dissemination of the implicit function theorem were (Peano and Genocchi, 1884) and Jordan 2nd edition of the *Cours d’analyse* (Jordan, 1893, pp. 80ff.). For Dini’s broader contribution to the foundation of real analysis see [Bottazzini, 1985].

²² [Tòth, 1972, 1980].

geometries had been outlined, with no mathematical (or physical) interpretation in terms of accepted objects and relations being given.²³

Gauss was apparently well aware that his differential geometry of surfaces might carry the potential to open a route towards such a missing interpretation, but he could not (or at least did not) solve the dilemma from a foundational point of view, that his surfaces were constructed inside the framework of Euclidean geometry. His reaction to Riemann's Habilitations lecture shows how well Gauss understood that Riemann had given a beautiful outline and far reaching program for another and much deeper conceptual step towards a trans-Euclidean geometry, which would reduce non-Euclidean theory in the sense of Bolyai and Lobachevsky to nothing but a special case. Riemann even sketched such a reduction in the last section of his talk, although he apparently had no knowledge of Bolyai's or Lobachevsky's studies in the foundations of geometry.²⁴

But the concept of manifold became essential for the understanding of non-Euclidean geometries in the late 1860-s and early 1870-s when the latter became finally absorbed into the general knowledge of mathematics. All three main contributors to non-Euclidean geometry in this phase, Beltrami, Helmholtz, and Klein, did refer to Riemann, whose Habilitations lecture became accessible to a wider scientific public outside Göttingen in 1867 after the publication in the *Göttinger Abhandlungen* (vol. 13). Here is not the place to discuss the role of Riemannian ideas in the development of knowledge and the discourse on non-Euclidean geometry in detail. It has to be said, however, that among the just mentioned authors, involved in the development of non-Euclidean geometry in the 1860-s, only Klein had been in contact with Riemannian ideas before he started to work on non-Euclidean geometry, through his close cooperation with A. Clebsch from 1866 onward. Beltrami and Helmholtz, in contrast, started to develop their ideas independently and progressed considerably before they learned to know of Riemann's lecture and adapted their presentation according to the latter's outlook. The shift in Beltrami's argument due to the influence of Riemann's view was particularly clear and seems to be characteristic for the broader turn geometry went through in the 1860-s and 1870-s and in particular to the role of the manifold concept in it.

E. Beltrami had started on his own in 1866 and 1867 to explore the possibilities inherent in the Gaussian theory of surfaces for an interpretation and understanding of non-Euclidean geometry. In early 1867 he realized that the geometry of the non-Euclidean plane can be gained in terms of a *generalized* Gaussian surface, i.e. the region

$$A = \{x \mid |x|^2 < a^2\} \subset \mathbb{R}^2$$

with metric not induced by an embedding in Euclidean 3-space, but "formally" given by

$$ds^2 = \frac{r^2}{(a^2 - x_1^2 - x_2^2)^2} ((a^2 - x_2^2) dx_1^2 + 2x_1x_2 dx_1 dx_2 + (a^2 - x_1^2) dx_2^2).$$

²³ The problematics of this type was addressed in Riemann's 1854 lecture by his opening remark, that earlier investigations on the foundations of geometry worked with purely "nominal" definitions. Although this remark was addressed at classical Euclidean definitions, Riemann hit a point which was even of higher importance for the contemporary status of non-Euclidean geometry, the discourse of which was apparently not known to him.

²⁴ Compare [Scholz, 1982a, pp. 220f.] and [Laugwitz, 1996].

He derived all properties essential for what would later be called the “Beltrami model” of the non-Euclidean plane, but insisted on the necessity to find a *real substrate* (*substrato reale*) of this “purely formally given” system in order to understand its geometric meaning. He was glad to find such a “real substrate”, by local isometric embeddings in classical Gaussian surfaces of constant negative curvature $\kappa = -r^{-2}$, embedded in Euclidean 3-space. So he sent a manuscript under the title *Saggio di interpretazione della Geometria non Euclidea* (published as (Beltrami, 1868a)) to Cremona as editor of the *Giornale di Matematiche*. Cremona disagreed with Beltrami’s narrow conception of “real substrate” of geometry, but nevertheless voted for publication after some period of hesitation and exchange with Beltrami about his views. Probably he doubted among others the mathematical value (not the correctness) of Beltrami’s observation that, although a “real substrate” could be given for the non-Euclidean *plane* by local isometric embeddings in classical Euclidean space, nothing similar could be hoped for in the case of three-dimensional non-Euclidean geometry (Beltrami, 1868a, p. 284).

After delivering the manuscript of his *Saggio* Beltrami got to know Riemann’s Habilitation lecture (maybe through a hint by Cremona) and changed his mind with respect to the epistemological (or even “ontological”) role of a classical interpretation for non-Euclidean concepts. He immediately prepared a second publication in which the two-dimensional case was generalized and an n -dimensional differential geometrical model for non-Euclidean geometry, using a simple Riemannian manifold representation, was given: $M \subset \mathbb{R}^{n+1}$ defined as a hemisphere, $|x|^2 = a^2$, $x_{n+1} > 0$, with metric induced by

$$ds^2 = (r^2/x_{n+1}^2) \sum_{i=1}^{n+1} dx_i^2$$

on \mathbb{R}^{n+1} . Parametrization of M by the open ball $|\tilde{x}|^2 < a^2$ with $\tilde{x} \in \mathbb{R}^n$ leads back to the case presented in the *Saggio* for the two-dimensional case.

Both articles appeared in the same year, although in different journals; Beltrami only made small adaptations in the text of the first one with general references to the possibility of a more conceptual understanding of non-Euclidean geometry than looking for a “real substrate”. The second article appeared as *Teoria fondamentale degli spazii di curvatura costante* (Beltrami, 1868b). The shift in interest and in outlook on the basic concepts of geometry between these two publications of Beltrami may serve as a concentrated expression for what was at stake in the change from classical geometry to modern geometry of manifolds. Beltrami lived through such a change in a couple of months, because his own line of thought already had brought him to the point of a formal generalization of Gauss’s theory of surfaces, and the inherent movement was so well dynamized by Riemann’s presentation.

Once Riemann’s construction of manifolds was accepted, even if only in the concrete version of “number manifolds”, the question of a “real substrate” for non-Euclidean geometry changed its meaning completely. To use later terminology, a differential geometric *model* of the metrically well explored (although from the axiomatic point of view still not completely elaborated) theoretical structure of non-Euclidean geometry could be given in a drastically extended framework. For the modern reader this extended conceptual framework has become so common that she may tend to overlook the hard work necessary to achieve the state of disciplinary practice and knowledge she is used to.

2.3. First appearance of manifolds in mathematical physics

Of course there are several semantical links of the manifold concept to physics, which could be pursued even in the 19th century. Riemann had already started to discuss such links on at least two levels. The final part and culmination of his Habilitations talk gave a sketch how in a subtle interplay between mathematical arguments and the evaluation of physical/empirical insights he proposed to come to a refined understanding of physical space. The essential bridge was an improved understanding of the microstructure of matter and its binding forces that should be, according to Riemann, as directly translated into differential geometric structures on manifolds as possible. But he also left the possibility open for further consideration that perhaps some time even a discrete structure of matter has to be taken into account, as it might very well be that the concepts of rigid body and light ray use their meaning in the small. Still, so Riemann argued by reference to astronomical measurements, the acceptance of a Euclidean space structure was well adapted to the physical knowledge of the time.

A second link was indicated in his famous Paris prize essay (Riemann, 1861/1876). Riemann there had modelled a three-dimensional heat flow problem in an ex ante inhomogeneous matter region and translated it into a differential geometric structure of a 3-dimensional Riemannian metric. In the result the question of a homogeneity criterion for the underlying matter could be analyzed as a question of local flatness of the metric. As is well known, that was the context in which Riemann published his most advanced results characterizing the curvature of a Riemannian manifold.²⁵ There should be no serious doubt, however, that Riemann was completely aware about the importance of such a connection between differential geometry and other parts of analysis or physics, although he did not, in the prize essay, elaborate explicitly on such a semantical connection, but motivated the interested reader to think along such lines by a highly interpretable reference to a Newton citation: “Et his principiis via sternitur ad majora.” (Riemann, 1861/1876, p. 391)²⁶

Recent historical investigations have shown how deeply connected large parts of the geometric discourse of the 19th century were to the semantics of physical space, even in parts of the discussion where, after the epistemological shift of mathematics brought about by the rise of set theory and the axiomatization movement at the turn of the century, a modern reader would not look for a direct semantical context in physical terms and would perhaps even tend to consider some parts of the debate at the end of the 19th century stricken by a surprisingly naive realism. This aspect has been discussed in detail by M. Eppele in his [1997] and shall not be reproduced in this article.²⁷ Of long-ranging interest for the development of higher dimensional manifolds in physics were, on the other hand, the first moves for a geometrization of state spaces in mechanics. This aspect has recently studied by J. Lützen, and my short report relies completely on his results.²⁸

²⁵ Compare among others [Reich, 1994; Laugwitz, 1996; Farwell and Knee, 1990; Scholz, 1980].

²⁶ Superficial and textpositivistic reading might give another picture of Riemann's intention. There are contributions to the historical literature like [Farwell and Knee, 1990] which deny the differential geometric content of Riemann's (1861/1876).

²⁷ Compare also M. Eppele's contribution in this volume for less “naive” attempts at physical semantics of topological concepts.

²⁸ Cf. [Lützen, 1988, 1995].

Most important among the geometrization arguments in this problem field were the following:

- (1) The subsumption of the least action principle for conservative systems under the form of a geodetical line. The state space was endowed with a physical metric of the form $ds^2 = 2(H - V) \sum g_{ij} dq_i dq_j$, for q_i coordinates in the state space, g_{ij} the metric induced on state space by the metric of the geometric coordinate space, H total energy, and V potential energy. That had been done analytically by Jacobi in the 1830-s and geometrized in low dimension ($n = 2$) by Minding, Liouville, and Serret about the middle of the century. Geometrization for higher dimensions was apparently discussed in the 1870-s and published, e.g., by Darboux in a particularly clear way in the year 1888.
- (2) Already a decade earlier Lipschitz developed a generalization of classical mechanics starting from a metric in the underlying geometrical space, which he allowed not only to be Riemannian but even Finsler (in modern terms) (Lipschitz, 1872). On that basis he developed a generalization of the term for the kinetic energy and the Hamilton–Jacobi form of mechanics. Moreover, in his discussion of conservative systems, he described the trajectories in the state space as (generalized) orthogonal to 1-codimensional submanifolds of the state space [Lützen, 1995, Section 49]. Other authors, not all of them aware of Lipschitz’ research, like Thomson/Tait and Darboux, pursued similar intentions.²⁹
- (3) The “Liouville theorem” on the volume preservation of the time flow in the phase space of Hamiltonian mechanics with canonically conjugate coordinates $q_i, p_i, 1 \leq i \leq n$, and dynamical equations

$$\frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$

which was presented by Liouville only in analytical formulation in a more general context (for the first time in 1838). Jacobi transferred it to mechanics by Jacobi about the middle of the century. Geometrization appeared in works on statistical mechanics only in the late 1860-s early 1870-s by Maxwell and Boltzmann (apparently without knowing about Liouville’s result).³⁰

- (4) Finally, Boltzmann’s discussion of different types of dynamical systems to characterize his idea of entropy contains a broad range of high-dimensional arguments in configuration or phase space, although in a highly intuitive manner. These are interesting questions for a broader history of the use of advanced mathematical concepts inside late 19th century physics, which are impossible to report here.

With respect to the claim made by Felix Klein in his famous historical lectures, that Gauss’s and Riemann’s differential geometry has supposedly “grown from up from the soil of the Lagrangian equations” [Klein, 1926/1927, p. 146], J. Lützen has shown in his detailed historical studies of the sources that this remark distorts history highly. Klein apparently did not allow for sufficient distinction between the original historical development (as documented and accessible from the sources) and his own perception of Riemannian

²⁹ Lützen considers Lipschitz’ generalized orthogonal trajectory discussion as the most important geometrization approach in mechanics during the 19th century [Lützen, 1995, Section 51].

³⁰ For more details see [Lützen, 1990, pp. 657ff.].

geometry that had formed in the late 1860-s early 1870-s, when he was a young mathematician and participated in the production of the events which he later told the history of. Explicit geometrization of configuration (state) and/or phase spaces of mechanical systems was in fact undertaken only in that relatively later period in which Klein was actively involved, and only then the language of higher dimensional manifolds became part of the discourse of theoretical physics and vice versa.

3. Steps towards a topological theory of manifolds

3.1. The 2-dimensional case as an elementary paradigm . . .

It was also in the 1860-s and early 1870-s that several lines of thought intertwined productively and led to the first relatively well-explored segment of a theory with links to different fields of study in the mathematics of the 19th century, the combinatorial theory of polyhedra,³¹ complex function theory, real projective algebraic geometry and the newly rising topological theory of manifolds. The main contributors to this subfield were A.F. Möbius (1863, 1865), C. Jordan with a series of publications through 1866, and Schläfli and Klein in their discussion on the orientability of real 2-dimensional subspaces of the projective space.³² In geometric function theory divers authors contributed to a refined understanding of the role of topological concepts, in particular C. Neumann with his calculation of the connectivity of a Riemann surface from the winding orders of branch points,³³ Lüroth, Clebsch and Clifford with their normalized representation during the 1870-s for branched coverings of $P_1\mathbb{C}$, which represent a Riemann surface with given number of leaves, given loci and winding numbers of branch points.

Möbius and Jordan both discussed independently from each other, which “morphisms” they wanted to consider for their topological theory of surfaces. Möbius called them “elementary relationships (Elementarverwandtschaften)” and Jordan just talked about “mappings”, and both circumscribed a transformation of “infinitely small elements” of one into the other, respecting neighbouring relations. They indicated that this idea could in principle be made precise by infinite series of subdivisions of the surfaces into finite surface “elements” which are one-to-one correlated, respecting the neighbouring relations.³⁴

Möbius gave in his article (1865) a detailed analysis of orientation procedures in surfaces, which he decomposed in polygonal nets (a generalized representation of a triangulation). He defined orientations of the boundaries of each polygon and coherence of neighbouring polygons, if the induced orientations in the common part of the boundaries are inverse to each other. As an application he gave the famous example of a non-orientable surface: a “Möbius band” complemented by a disc to form a closed non-orientable surface homeomorphic to $P_2\mathbb{R}$ (which Möbius did not explicitly remark) (Möbius, 1865, p. 483).

His earlier publication on “elementary relationships” contained a topological classification of closed orientable surfaces embedded in \mathbb{R}^3 (without self-intersection). He classified singular points of a “height” function geometrically into “elliptical” and “hyperbolic”

³¹ See [Lakatos, 1976].

³² For Schläfli–Klein compare Section 2.1.

³³ Genus p of the surface given by $2p = \sum_{i=1}^k (m_i - 1) - 2n - 2$ for a Riemann surface with n leaves over $P_1\mathbb{C}$ and k branch points of orders $m_i - 1$ (Neumann, 1865).

³⁴ More details in [Pont, 1974] or [Scholz, 1980, pp. 148ff.].

points and developed what from a 20th century point of view reads as a geometric presentation of the Morse theory of differentiable closed orientable surfaces. He showed that each such surface F can be constructed from two homeomorphic (“elementary equivalent”) surfaces F_1 and F_2 , each with exactly n boundary components, which are pasted together at the boundary components. Möbius called n the “class” of the surface and showed that it was a classifying invariant. He did not remark, however, that his “class” and Riemann’s “genus”³⁵ p were essentially the same, with $p = n - 1$.

While there is no indication that Möbius knew the function theoretic work of Riemann and its topological aspects, he did connect his studies to the famous debate on Euler’s polyhedral formula and proved it in the general case using his invariant, $\chi(F) = 2(2 - n)$. In our eyes it reads of course more naturally if rewritten with Riemann’s invariant, $\chi(F) = 2 - 2p$.

Jordan classified orientable surfaces, including those with boundary, independently from Möbius. He counted the maximal number k of recurrent cuts (cuts along double-point-free pairwise disjoint closed curves c_i , $1 \leq i \leq k$), which do not dissect the surface into disconnected pieces, and the number m of boundary components. He showed that the pair (m, k) classifies the orientable (compact) surfaces uniquely (Jordan, 1866, p. 85). For the proof he used dissection of the surfaces along the recurrent cuts and additional cross cuts and topological maps of the resulting simply connected pieces.

Jordan, in contrast to Möbius, was aware of the connections between the topological theory of surfaces and complex function theory. Another aspect of his work on surfaces, the study of homotopy classes of closed paths, very likely was motivated by this context, although he did not remark so explicitly and left it to the reader to realize it. Riemann had been inspired in his topological investigations of surfaces by the behaviour of the integrals of holomorphic differential forms and thus considered a homological equivalence concept between closed paths (cycles); but of course in complex function theory the question of analytic continuation and the resulting questions of multi-valuedness played an important role (including Riemann’s work, as is well known). For analytic continuation the continuous deformation of paths, or in later terminology a homotopic concept of equivalence between cycles was the proper one to study. Jordan did not explain this, but he gave a complete description of the homotopy theory of his bounded orientable surfaces, including definition of the equivalence concept, generators and relations of the fundamental group. This beautiful and surprising aspect of Jordan’s work is discussed in detail in Vanden Eynde’s contribution in this volume and therefore not documented in more detail here.

Here I only want to repeat that Jordan did not use explicit group terminology, as the group concept was in the middle of the 1860-s still essentially confined to substitutions. He nevertheless must have been aware of a conceptual relationship between what he did with the deformation classes of closed paths and groups, as he had been actively involved in Galois theory in the time immediately before.

Taken Riemann’s, Möbius’, and Jordan’s work together, and perhaps adding Schläfli and Klein, it becomes clear that at the transition from the 1860-s to the 1870-s a complete topological theory, including classification, homology and homotopy aspects for compact orientable surfaces was at hand and widely accessible, and a first elaboration of questions of non-orientability had been started. Betti, moreover, had indicated how a generalization

³⁵ The terminology “genus” is due to Clebsch (1864).

of the homological part of the theory to higher dimensions might work; although the route he had indicated was still unexplored.

3.2. . . . and first attempts to understand higher dimensions

In 1871 Betti published his presentation of higher numbers of connectivity. The objects of his study were n -dimensional submanifolds S_n of an \mathbb{R}^m , *spazi*, as he called them, in general supposed to be closed and connected.³⁶ The method of characterizing connectivity numbers (“Betti numbers”) p_k ($1 \leq k \leq n$) was to consider maximal systems of closed k -dimensional submanifolds, U_i ($1 \leq i \leq p$), which “cannot form the border of a pathwise connected (sic!) $(m + 1)$ -dimensional part of the space” (Betti, 1871, p. 278).

Like Riemann in his (not yet published) fragment Betti argued that the maximal number p is independent of the choice of the system of submanifolds, using step-by-step substitution of the cycles. His verbal description of the boundary relation was, however, not precise enough to exclude counterarguments, which were given by Tonelli (1875) showing that a more refined symbolism for the representation of the cycles and their homology relations was needed. Moreover, Tonelli corrected the unnecessary and for the argument detrimental specification of pathwise connectedness for the bounding part of the surface. These necessary criticisms did not lessen Betti’s achievement of a public presentation of the first step towards a homological theory of manifolds, which until then had lain latent in the thought and manuscripts of Riemann and some (provably at least one) of the latter’s closest correspondents.

There remained the lacuna, however, that although the method was presented for n -dimensional (closed) manifolds in general, no new insights were immediately accessible by this method for higher dimensions with the exception of the simplest three-dimensional examples. Betti, e.g., discussed the connectivity of the “thickened” two-sphere and the massive and the “thickened” torus in \mathbb{R}^3 in letters to P. Tardy written in 1863, although published only in 1915 (Betti, 1915). It nearly remained so until Poincaré’s great series on analysis situs at the turn of the century. There was, however, at least one other intermediate step of long standing significance, E. Picard’s investigation of the topology of complex algebraic surfaces at the end of the 1880-s and in the early 1890-s.

Picard combined with great imagination ideas from algebraic geometry, complex analysis, early homology and homotopy to analyze the topological structure of algebraic curves. He noticed in the early 1880-s, as M. Noether had done already a decade earlier³⁷ that in algebraic surfaces integrals $\int_c \omega$ of meromorphic differential forms without first order poles (forms of first or second kind) over 1-dimensional cycles c are 0 “in general” (i.e. for most algebraic surfaces). Picard gave a detailed explanation of this phenomenon by an analysis of the first Betti number of a generic algebraic surface F . Starting from a singularity-free birational model of F in $P_3\mathbb{C}$ he derived a representation in projective three-space such that the resulting equation for F , $f(x, y, z) = 0$ (in inhomogeneous coordinates), leads to a 1-parameter family F_y of algebraic curves, which, with the exception of a finite set of values $Y = \{y_1, \dots, y_k\}$, are of the same genus p . From the topological point of view Picard thus studied a fibration $F \rightarrow P_1\mathbb{C}$ with a closed oriented surface of genus p as

³⁶ Compare Section 2.1.

³⁷ (Noether, 1870, 1875) and (Picard, 1885, p. 282; 1886, p. 330).

generic fibre and a finite set of exceptional fibres with genus $< p$. By a beautiful blend of complex analytic and topological arguments, combining homotopy classes of closed paths in $P_1\mathbb{C} \setminus Y$, homology classes of 1-cycles in a generic fibre F_y , abelian integrals and the monodromy of the “Picard–Fuchs” differential equation describing the change of values of abelian integrals under change of y , he showed that in the generic case (for “most” F) all 1-cycles reduce homologically to only one already by monodromy constructions of boundary relations. Then by the observation that in each singular fibre F_{y_i} at least one cycle degenerates to a point, he argued convincingly that such a “vanishing cycle” is homologically trivial in F , and thus all 1-cycles are homologically zero. Picard started to use the same arsenal of methods to calculate the second Betti number of F , but did not get far in this attempt. Apparently the symbolical methods were not sufficiently elaborate to deal with this more involved situation before Poincaré entered the arena.

4. Passage to the theoretical stage

4.1. Poincaré entering the field

During the 1880-s Poincaré came across “manifolds” in several analytical or geometrical contexts, although he personally understood them at that time still in a rather vague way. One of these contexts arose from his work in the theory of automorphic functions that made him famous (and Klein nervous) at the beginning of the decade. The culminating problem of Poincaré’s and Klein’s research was the uniformization “theorem”. Poincaré’s points of departure were complex differential equations over algebraic curves

$$\frac{d^2v}{dx^2} = v(x)\phi(x, y),$$

ϕ being a meromorphic function on an algebraic curve C given by $f(x, y) = 0$, which leads only to (finitely many) regular singularities.³⁸ If it could be solved by means of a pair of Fuchsian functions³⁹ $x(\zeta)$, $y(\zeta)$ taking as fundamental system the functions $v_1 = \sqrt{dx/d\zeta}$ and $v_2 = \zeta\sqrt{dx/d\zeta}$ (pushed down to C), Poincaré called the equation a *Fuchsian* differential equation.⁴⁰ The quotient $v_2/v_1 = \zeta$ was then the inverse of a universal covering map of the algebraic curve C , branched in the (regular) singularities of ϕ on C . Poincaré called two Fuchsian equations of the same *type* if there is a birational transformation between the underlying algebraic curves C and C' which transforms the singularities one into another such that the monodromy characteristics remain identical.⁴¹

For the sketch of a proof Poincaré (1884) collected all types of differential equations on an algebraic curve of given genus p and with given number k of branch points and

³⁸ p is *regular singular point* of the differential equation if a fundamental system of solutions can be chosen such that the quotient is a multivalued function branching over p of order k ($k \in \mathbb{N}$) or ∞ .

³⁹ That is, ζ varies in the Poincaré half plane $\text{Im}(\zeta) > 0$ and x, y are invariant under a properly discontinuous subgroup $G \subset PSL_2(\mathbb{R})$.

⁴⁰ For Fuchs’s studies of the monodromy of regular singular differential equations compare [Gray, 1984; 1986, pp. 60ff.].

⁴¹ That means, the difference of the characteristic exponents of two fundamental solutions of the equation is identical and of the form $1/k$ or 0 with $k \in \mathbb{Z} \setminus \{0\}$. In that case the quotient ζ is the inverse of a branched covering of branching order $|k|$ or ∞ .

branching orders l_i ($1 \leq i \leq k$) in a “multiplicité” M which in generic points could be characterized by $6p - 6 + 2k$ (real) parameters. Analogously he parametrized the Fuchsian groups which lead to the proper genus p and given branching behaviour in another “multiplicité” M' (of the same dimension). Poincaré’s version of the *uniformization theorem* then claimed that each type contains at least one Fuchsian differential equation.⁴² To argue for the correctness of this claim he considered the map $g : M' \rightarrow M$ and showed that it is continuous and injective. The main point of the famous “continuity” proof was then to conclude the surjectivity of g from this information.

Poincaré gave a discussion which in fact spoke in favour of the surjectivity and was already sharper than Klein’s, but still used highly intuitive ideas about continuous variation of images in higher dimensional spaces in a symbolically uncontrollable manner. Even the spaces themselves were not shown to be manifolds but taken as such, without further ado. For any critical reader (perhaps even including Poincaré himself) the “continuity proof” could thus be taken at least as much as an indicator for the necessity of an improved understanding of higher dimensional geometry as it was an indicator for the truth of the uniformization theorem. And in fact a clarification of the topological proof strategy was given only later by Brouwer (1911a, 1911b) who used enlarged (necessarily no longer uniquely) parametrizing spaces which indeed were manifolds and to which he could apply his domain invariance theorem for continuous injective mappings.⁴³

Another context in which Poincaré gathered early experiences with higher dimensional manifolds arose from his investigation into the qualitative theory of differential equations. One of his questions was the topological classification of the singular points of a vectorfield (node, saddle point, focus, centre) and the introduction of the index as a numerical invariant. After having done so in the plane, he modelled nonlinear ordinary differential equations by the flow of a vectorfield v on (real) algebraic surfaces F and realized that the global index of the vectorfield $\text{ind}(v)$ (i.e. the sum of the local or pointwise defined indexes) is equal to the Euler characteristic of the surface: $\text{ind}(v) = 2 - 2p$ (Poincaré index theorem) (Poincaré, 1885).⁴⁴

In an attempt to generalize the result to nonlinear differential equations of higher order he transformed that problem to a high dimensional system of first order equations. Then he started with a geometrization of the n -dimensional situation, although at the outset he considered geometrization as nothing more than a “useful language” (Poincaré, 1886, p. 168).

Fortunately he could build upon results of Kronecker (1869) about an analytically defined concept of index of functions systems on hypersurfaces of \mathbb{R}^n , which about the same time was being given a topological content by W. Dyck.⁴⁵ Dyck had shown that the Kronecker characteristic could be expressed in terms of his own purely topologically defined characteristic, which in fact was equivalent to the Euler characteristic and even equal up to sign.⁴⁶ Working with the Kronecker characteristic as a symbolical tool,⁴⁷ Poincaré was

⁴² In Poincaré’s terminology: Each type is a “Fuchsian” type.

⁴³ On the other hand, Brouwer made it clear that the moduli spaces used by Klein and Poincaré had singular points in curves with a nontrivial birational automorphism group, so that the argument of Poincaré and Klein turned out in fact to be unreliable in its original form.

⁴⁴ Cf. [Gilain, 1991; Gray, 1992; Dahan, 1997].

⁴⁵ Dyck published his first results on topological characteristics in the years 1885–1887 in the *Mitteilungen Sächsischer Gesellschaft der Wissenschaften*.

⁴⁶ [Scholz, 1980, pp. 249ff.].

⁴⁷ Poincaré did not cite Dyck, whose publication he apparently did not know.

able to sketch the idea of a high dimensional version of the index theorem for vector fields, the later Poincaré–Hopf index theorem,⁴⁸ for the case of open submanifolds or hypersurfaces in \mathbb{R}^n . In this context Poincaré explicitly expressed the need for further elaboration of the methods to determine the higher orders of connectivity of Riemann and Betti (Poincaré, 1886, p. 448).⁴⁹

Of course, there were other contexts in which Poincaré found an opportunity to come back to manifold ideas, for example in his studies of complex integration in two variables $\int f(\xi, \eta) d\xi d\eta$ with $(\xi, \eta) \in \mathbb{C}^2$. Poincaré showed that (in modern notation) $d\omega = 0$ for $\omega = f d\xi \wedge d\eta$ and therefore the Cauchy theorem holds for two-dimensional integrals. Interestingly enough he still used the traditional language of “deformation” of one surface S into another S' through a region $A \subset \mathbb{C}^2$, in which the 2-form is analytic, although from the context it must have been clear to him that a homological concept of boundary relations was closer to the situation (Poincaré, 1887, p. 456). In his earlier experiences with high dimensional geometry Poincaré had been skeptical with regard to its usefulness, as he argued that spatial intuition would no longer be directly applicable. By the late 1880-s however, he had gathered sufficient material in different fields of his studies for him to accept that such “hypergeometrical” language of “multiplicités” are useful and perhaps even necessary for proceeding further with some of his analytical investigations.

4.2. A constructive approach to manifolds

The exclusion of a direct application of spatial intuitions would not exclude indirect application, mediated by a proper symbolical framework, which had been only roughly sketched by Riemann and Betti. That is what Poincaré started to pursue in the early 1890-s and continued to work on for the rest of his life, best expressed in his ground breaking series of articles on “analysis situs” (Poincaré, 1895, 1899, 1900, 1902a, 1902b, 1904). In this series Poincaré set the stage for a theoretical exploration and characterization of manifolds of any (finite) dimension which expanded so fruitfully and vastly in our century. Moreover, in the elaboration of the tools of analysis situs to make the “hypergeometry” of manifolds symbolically accessible, he brought combinatorial topology to the point where it could easily transcend the limits of manifolds and become a field of study of its own. Some traits of the theoretical and methodological achievements are outlined in the next section.

Poincaré, in accordance with his general philosophy of mathematics, did not use a formal, perhaps even axiomatic, definition of manifolds (which would moreover have been rather difficult to formulate in the 1890-s), but preferred to outline constructive procedures for the generation of manifolds. He used two main procedures to define a manifold M .

- (1) In his first definition he described M as zero set $f^{-1}(0)$ of a differentiable function $f : A \rightarrow \mathbb{R}^k$, with A open subset in \mathbb{R}^{n+k} , defined by inequalities, and the Jacobian $df(x)$ of maximal rank for all $x \in A$. He admitted that M might have a *boundary*. More clearly than his predecessors Poincaré explicitly used the rank condition to derive local parametrizations of M . In addition he explained the morphisms under which two such representations M and M' should be considered as equivalent, as

⁴⁸ (Hopf, 1926). An intermediate step was taken by Brouwer in his work on the index of vectorfields on n -dimensional spheres (Brouwer, 1911b, pp. 107ff.); cf. [Johnson, 1987, p. 82].

⁴⁹ Ironically Poincaré got the name wrong speaking about “Brioschi” where he obviously referred to Betti’s work (Poincaré, 1886, p. 448), showing that he just started to assimilate the subject.

diffeomorphisms of open neighborhoods of M , respectively M' , in their embedding real space, which map M onto M' or vice versa. In his terminology he did not even indicate the possibility of a distinction between a general topological and a differential topological structure, although he used the terminology of “homéomorphismes” (Poincaré, 1895, pp. 196ff.). In fact Poincaré alluded to Klein’s Erlanger program and called the groupoid of his diffeomorphisms a “group”, which should define the branch of geometry called “analysis situs”, as he saw it (ibid., p. 198).

- (2) The second main definition allowed for a finite set (we would say atlas) of differentiable regular parametrizations of M by domains V_i in \mathbb{R}^n . Poincaré considered M as covered by sets U_i which all were subsets of \mathbb{R}^m ($m \geq n$), without considering M as a subset of \mathbb{R}^m : $M = \bigcup_{i \in I} U_i$ with parametrizations $\Theta_i: V_i \rightarrow U_i$ ($1 \leq i \leq l$) and regular change of parametrization in n -dimensional components of overlaps of the U_i -s (and with a similar definition in the complex case).⁵⁰ Poincaré concentrated attention in this definition on the analytic case and used the terminology of “analytic continuation” for the description of change of parameters in overlapping regions (Poincaré, 1895, p. 200). In this case he introduced the *orientability* of M by the condition of positive functional determinant for changes of parametrization.

Of course, Poincaré did not exclusively consider these main definitions, but explained how to derive local parametrizations from the first definition, discussed diverse examples of mixed constructions, e.g., by restriction of the parameter sets V_i to lower dimensional submanifolds of the parameter space \mathbb{R}^n , defined by method (1). Even the operation of a finite group leaving the parametrizing submanifold invariant was included, as in his description of an image of the real projective plane $P_2\mathbb{R}$ in \mathbb{R}^6 .⁵¹

Depending on the context of investigation, Poincaré later introduced additional construction procedures, which presupposed that the resulting object satisfied definitions (1) or (2).

- (3) The most important of these additional constructs was the cell subdivision and the representation of M as a finite geometric *cell complex* a “polyèdre”, which by definition satisfies the local manifold condition (Poincaré, 1895, pp. 270ff.). He used it inter alia for constructing manifolds with prescribed fundamental group by boundary identification rules, although restricted to the three-dimensional case, where the local manifold conditions about identified 0-cells could be controlled by a combination of symbolic representation and basic space intuition (Poincaré, 1895, pp. 229ff.).⁵²
- (3') In the fifth complement (1904) Poincaré introduced even more construction procedures, a “skeleton” representation of 3-dimensional manifolds, containing some ideas of three-dimensional Morse theory (Poincaré, 1904, pp. 475ff.), and an adaptation of an idea of P. Heegard to form a closed 3-manifold M by boundary identification of two homeomorphic handle bodies V and V' .⁵³ He used these procedures

⁵⁰ Poincaré would not read M as literal union of the U_i . In lower dimensional components of intersection $U_i \cap U_j$ he thought in terms of a disjoint union, only in n -dimensional components he would identify the respective points of U_i and U_j ; therefore he did not treat M globally as subset of \mathbb{R}^m . In more recent terminology, M is a manifold, while the immersion used in Poincaré’s construction is not necessarily injective, and thus the image \tilde{M} no (sub-) manifold of \mathbb{R}^m .

⁵¹ Poincaré used a parametrization of $P_2\mathbb{R}$ by $S^2 \subset \mathbb{R}^3$ with antipodal identification.

⁵² These examples are discussed in detail in [Volkert, 1994, pp. 87ff.]; compare also [Volkert, 1997].

⁵³ Compare [Volkert, 1994, pp. 137ff.].

to present the famous Poincaré “dodecahedral space” M with trivial homology but fundamental group isomorphic to the (extended) dodecahedral/icosahedral group I^* (Poincaré, 1904, pp. 478ff.).

Poincaré was convinced that each manifold given by definitions (1) or (2) can be represented as a (finite) “polyèdre” as in definition (3). His arguments in favour of that conviction were, however, more founded on intuitive “optimism” than on critical evaluation of the question (Poincaré, 1899, pp. 332ff.). So Poincaré claimed to have a positive solution for what later was considered to be a basic problem for the clarification of the conceptual structure of the topological theory of manifolds, the *existence of triangulations* for differentiable or topological manifolds.⁵⁴ A similar evaluation can be given for his use of the principle that it is always possible to find a common subdivision of two given finite cell subdivisions of a manifold (Poincaré, 1895, p. 271), which later became the *Hauptvermutung* in the terminology introduced by Kneser.⁵⁵

Already from this short presentation it may become apparent that Poincaré’s constructive concept of manifold contained an arsenal of methods to build examples to enrich the understanding of the world of new geometric objects. Although he did not even attempt to give a formal analysis and unified delimitation of the concept, Poincaré’s work was thus highly effective and gave a tremendous push towards a more refined understanding of the general concept outlined by Riemann and so difficult to understand in the second half of the 19th century.

4.3. Giving a theoretical status to the topology of manifolds

These examples of manifolds constructed and considered by Poincaré served as material for the exploration and development of methods to analyze their intrinsic *analysis situs* nature. Poincaré’s work is, of course, much better known by its contribution to these methods than by the elaboration of the basic material of manifolds.⁵⁶ In fact, Poincaré presented two approaches to analyze the homology of manifolds, the first followed Riemann and Betti rather directly and was introduced in the opening work of the series (Poincaré, 1895). The second one with a presentation and elaboration of the homology of cell complexes was the subject of the first two complements (Poincaré, 1899, 1900). Moreover he introduced the fundamental group of manifolds already in (Poincaré, 1895) and constructed diverse examples of 3-dimensional manifolds with prescribed fundamental group. These more elementary examples were superseded by the elaborate case of the “dodecahedral” space in the fifth complement (Poincaré, 1904). In the two intermediate supplements he developed methods to calculate the homology of algebraic surfaces (Poincaré, 1902a, 1902b). Diverse detailed historical studies deal with different aspects in Poincaré’s topological work;⁵⁷ here I only want to outline the homological part of the profile of the theory which Poincaré pro-

⁵⁴ Compare [Kuiper, 1979].

⁵⁵ Compare [Volkert, 1994, p. 164] and other contributions in this volume.

⁵⁶ For the latter aspect see [Volkert, 1994].

⁵⁷ For the homological aspects of Poincaré’s work consult primarily [Bollinger, 1972] and in addition [Dieudonné, 1989, 1994] and from a semiotic point of view [Herrmann, 1996], for the homotopic aspects [Vanden Eynde, 1992], for specific construction methods of low 3-dimensional manifolds [Volkert, 1994], for a discussion of the contribution to the manifold concept and an outline of Poincaré’s topological study of algebraic surfaces [Scholz, 1980]. Compare also diverse other contributions to this volume.

posed in order to make the “hypergeometry” of manifolds accessible. Poincaré’s introduction of the fundamental group is discussed in the article of Vanden Eynde (this volume).

In his first approach to homology in a n -dimensional manifold M Poincaré followed Riemann’s proposal to study bordance relations of *oriented* submanifolds V_1, \dots, V_k of given dimensional $m \leq n$. His procedure was conceptually still highly intuitive and vague, as the underlying idea supposed the study of equivalence relations on the set of all m -dimensional submanifolds and was too complicated to get hold of, with the methods available at the time. Moreover, to make the approach feasible, the “submanifolds” should admit certain “nicely behaving” singular subsets, like the topological (or smooth) “varieties” recently proposed by (Kreck, 1998). Poincaré, however, went a step further than his predecessors in the symbolical description of his objects and relations. In particular, he introduced an algebraic representation,

$$V_1 + V_2 + \dots + V_k \sim 0,$$

for the condition that all the V_i form a complete boundary of an $(m + 1)$ -dimensional submanifold and transformed such homology relations by addition and multiplication of the terms with integer coefficients. In this approach the terms were of a peculiarly ambivalent semiotic nature. Basically, Poincaré interpreted terms like λV_i ($\lambda \in \mathbb{Z}$) as a collection of λ “slightly varied” copies of the (oriented) submanifold V_i ; but he accepted and used a formal division of homologies,⁵⁸ as a result of which the homologies no longer *directly* had to express boundary relations.⁵⁹ In the result Poincaré got an interesting symbolical system for homologies and the calculation of Betti numbers p_i ,⁶⁰ which allowed him to explore basic features of the homology of manifolds much deeper than his predecessors, in particular duality for the Betti numbers, $p_i = p_{n-i}$ for orientable closed manifolds, and the Euler–Poincaré theorem $\chi(M) = \sum_{i=0}^n (-1)^i p_i$.

None of these could be proven indubitably in Poincaré’s approach. For the duality theorem his calculation of the intersection numbers remained highly intuitive, as the differential topology of general transversal intersections was too involved to be clarified by his means. For the generalized Euler theorem Poincaré used his principle of the existence of a common refinement of two finite cell decompositions of the manifold M (the later *Hauptvermutung*). So, from a critical point of view, both principles (Poincaré duality and Euler–Poincaré) had rather the status of well motivated conjectures than of “theorems”, even in the eyes of critical contemporaries like Heegard, Dehn, Tietze et al.

After Heegard’s criticism of the discussion of duality in manifolds, Poincaré established his second, much better algebraicized combinatorial method to define and calculate connectivity numbers, adding torsion numbers and coefficients to the Betti numbers (Poincaré, 1899, 1900).⁶¹ He started from a representation of the manifold M as a geometric cell complex constituted by q -cells $a_i^{(q)}$ ($1 \leq i \leq \alpha_q$ for all dimensions $0 \leq q \leq n$), and described

⁵⁸ He made the “division rule” explicit in the first complement answering P. Heegard’s criticism of having suppressed torsion elements. In (Poincaré, 1895) it was subsumed under a sort of “metarule” for homologies: “Les homologies peuvent se combiner comme des équations ordinaires” (Poincaré, 1895, p. 207).

⁵⁹ Compare the often discussed example of the line l in $P_2\mathbb{R}$ with $2l \sim 0$ having a direct geometric interpretation as a small angular segment U between two lines, $\partial U = 2l$; whereas the result of division $l \sim 0$ had no longer direct geometric interpretation, as criticized by Heegard.

⁶⁰ To be precise, Poincaré used a slightly changed definition of Betti numbers $P_i := p_i + 1$, if p_i is the maximal number of homologically independent i -cycles (“with division”, i.e. calculating with integer coefficients in \mathbb{Q}).

⁶¹ Compare [Bollinger, 1972].

boundary identifications as “congruences”, $a_i^{(q)} \equiv \sum_j \varepsilon_{ij}^{(q)} a_j^{(q-1)}$, codified by the matrices $E^{(q)} = (\varepsilon_{ij}^{(q)})$, and reduced the consideration of cycles and their boundary relations to those expressible in linear combinations of cells. That allowed him, of course, to avoid the difficult problems arising from investigation of all submanifolds and led to the well-known approach of combinatorial topology. Poincaré could thus very well show that the most evident difficulties arising from his first approach resulted geometrically from nonorientability of the manifold M and algebraically from the ambivalence between homology “with division” (we would say calculating the homology over \mathbb{Q}) and “without division” (over \mathbb{Z}).

Poincaré (1900) presented a new definition and a calculus for the calculation of Betti numbers and torsion from the incidence matrices $E^{(q)}$ of a cell decomposition of M . The method used diagonalization of incidence matrices by elementary transformations to matrices $T^{(q)}$. Expressed in slightly more structural terms Poincaré developed a calculus to choose generators of the \mathbb{Z} -module C_q of cellular q -chains such that all boundary operators $\partial_q : C_q \rightarrow C_{q-1}$ are diagonalized. That allowed him to read off immediately the Betti numbers and torsion coefficients and the distinction between manifolds “with” or “without” torsion from the diagonalized matrices $T^{(q)}$ (Poincaré, 1900, p. 369).

Poincaré was sure that his second, the combinatorial, method led to the same homological invariants (Betti numbers and torsion coefficients) as the first, bordance of submanifolds, method. He first showed that a subdivision of the “polyédre” does not change the combinatorial invariants (1899, pp. 303ff.). Considering now a set of submanifolds, arising in the representation of cycles and homologies of the first method from the “principle” of the existence of cell subdivision for each of them and the assumed possibility of constructing a common subdivision (“Hauptvermutung”), he concluded without any hesitation that the “old” and the new (combinatorially defined) homological invariants are identical. This part of the “proof” needed only *six lines* in his presentation (Poincaré, 1899, p. 309). Although he thus got new problems he could not solve or even realize, he achieved on the other hand a proof of duality for orientable manifolds and the generalized Euler theorem in the symbolically clear framework of the new approach (1899, pp. 302f.; 313ff.).

In the end Poincaré had achieved a lot for a homological theory of (differentiable compact) manifolds about the turn of the century. He had introduced the old invariants (Betti numbers) in a new, much clearer symbolical framework, had introduced new ones (torsion coefficients), developed a well algebraicized calculus to compute them, calculated them in a great variety of cases, and proven two basic theorems (duality, Euler–Poincaré). Moreover he had introduced and given a basic analysis of the topological importance of the fundamental group, which is put into the context of the development of homotopy ideas in the contribution of R. Vanden Eynde in this volume. Thus, even taken into consideration that Poincaré took basic principles to be valid without any hesitation (triangulability, Hauptvermutung), that turned out to contain serious problem potential for the future clarification of basic structures of the topology of manifolds during the century to come, there can be no doubt that he was the main initiator of a topological theory of manifolds of wide range.⁶² Moreover, the elaboration of his second (combinatorial) approach to homology opened the path towards a homological theory of more general topological spaces.

⁶² This advancement tends to be suppressed in Dieudonné’s discussions of Poincaré, as he looks at the latter rather with the eyes of a “modern” mathematician in the sense of the 20th century than with those of a historian.

5. Elaboration of a logical frame for the modern manifold concept

5.1. Early axiomatic attempts for two-dimensional manifolds

Topological spaces on different levels of generalization were analyzed in different approaches and with varying degrees of precision in the rise of modern mathematics in the early 20th century. During the last three decades of the 19th century Cantor had developed his theory of point sets in \mathbb{R}^n in the framework of general set theory. He himself was shocked to realize that bijective maps between real continua of different dimensions can be conceived, and even Dedekind's comforting conviction that more specific maps, in this case bijective and (bi-)continuous ones, would respect the *invariance of dimension* left the problem to prove (or disprove) such a conjectured invariance. Naive assumptions from space intuition were particularly deceptive in this field; that became even clearer about 1890 when Peano published his example of a "spacefilling" curve with the surprising effect, that the lack of injectivity would even for continuous maps not necessarily lead to a decrease of dimension (or keep it at most invariant), but could as well *increase* it. Early attempts by Lüroth, Thomae, Netto, and Cantor himself, to prove the invariance of dimension under bijective continuous maps, turned out to contain unclosable gaps and again (as in the case of the continuity proof for uniformization) it was only Brouwer who surmounted the difficulties and indeed proved the correctness of Dedekind's suggestion (Brouwer, 1911a).⁶³ About the turn of the century two methodological strategies for clarifying the concept of manifold were formed and sketched, an axiomatic one proposed by Hilbert, taken up by Weyl (about 1913), Hausdorff, H. Kneser, and Veblen/Whitehead, and a constructive one proposed by Poincaré, taken up by Dehn/Heegard, Tietze, Steinitz, Brouwer, Weyl (after 1920), Vietoris, van Kampen and others.

The first attempts at an axiomatic formulation of manifolds by Hilbert and by Weyl (1913) were limited to dimension 2 by contextual considerations. They contained a blend of early ideas of general topology and postulates for regular coordinate systems as specific manifold structures. Hilbert's approach (1902a, 1902b) arose from the context of the foundations of geometry and had as its main goal the erection of an axiomatic framework for the concept of a (simply connected) two-dimensional continuous manifold which should serve as a starting point for a group theoretic characterization of the principles of Euclidean geometry.

Hilbert supposed the *plane* E to be topologized by a sufficiently rich system of *neighbourhoods* ("Umgebungen") \mathcal{U}_p of each point $p \in E$, formed by sets $U \subset E$ containing p and each complemented by at least one coordinate bijection $\psi : U \rightarrow V$ onto a Jordan domain $V \subset \mathbb{R}^2$, such that the four following conditions hold:

- (1) For each Jordan domain $V' \subset V$ containing $\psi(p)$ the counterimage $\psi^{-1}(V')$ is also a neighbourhood of p .
- (2) For two coordinate bijections ψ and ψ' of the same neighbourhood U onto V and V' the coordinate change $\psi'\psi^{-1} : V \rightarrow V'$ is bijective and continuous.
- (3) A neighbourhood V of $p \in E$, containing a point q , is also a neighbourhood of q .
- (4) Each two neighbourhoods V, V' of p contain another one $V'' \subset V \cap V'$.
- (5) To any two points $p, q \in E$ there exists a common neighbourhood V .

⁶³ For the history of invariance of dimension see [Johnson, 1979/1981, 1987] and for an outline of Brouwer's proof [Koetsier and van Mill, 1997].

Hilbert commented that his postulates contain, as he thought, the “precise definition of the concept, which was called *multiply extended manifold* by Riemann and Helmholtz and *number manifold* by Lie” (Hilbert, 1902a, p. 233). This remark of Hilbert was, like so many others in the foundations of mathematics, a bit rash but showed a promising way to proceed.

In fact, Hilbert’s sketch of an axiom system for two-dimensional manifolds containing all the conceptual components for the later refinement of both, the characterization of general topological spaces, by what would later be called a neighbourhood basis as formulated by Hausdorff (1914)⁶⁴ and the axiomatic definition of manifolds by coordinate systems and a regular atlas as elaborated by Veblen and Whitehead (1931). Hilbert dealt, however, with both aspects in a simplified form justified by his restricted context. The later Hausdorff separability was indirectly implied by his last axiom of “big” coordinate neighbourhoods to any two points $p, q \in E$ and their separability in the coordinate plane by Jordan regions (in addition to axiom (1)).

Weyl, in his *Idee der Riemannschen Fläche*, could already build upon Brouwer’s result of the invariance of dimension under bijective continuous maps between open sets in \mathbb{R}^n . That may have given him the confidence that a slightly more “intrinsic” characterization (than Hilbert’s) of a two-dimensional manifold was possible.

Like Hilbert he characterized the structure of a two-dimensional manifold F by a system of neighbourhoods U_p , each of which, $U \subset F$ would contain p and be supplemented by a bijective map $\psi : U \rightarrow V \in \mathbb{C}$, with V an open disk with center $\psi(p)$. The totality of neighbourhoods was used by Weyl as a neighbourhood basis for the topology of F in the modern sense. He demanded that they satisfy two conditions. The first one amounted to what would be expressed in more recent terminology as (i) the *open map* condition for the coordinate map ψ with respect to the topology on F induced by the neighbourhood basis. The second one was: (ii) for any neighbourhood U of a point $p \in F$ with coordinate map $\psi : U \rightarrow V$ and a small disk $V' \subset V$ of center $\psi(q)$ ($q \in U$), there is a neighbourhood U' of q such that $\psi(U') \subset V'$.

The second postulate had a double function in Weyl’s argument; it made sure that coordinate maps were continuous and it secured the existence of sufficiently many “neighbourhoods” to constitute a neighbourhood basis (from our point of view). Essentially Weyl characterized a manifold F as a topological space by the assignment of a neighbourhood basis \mathcal{U} in F , postulating that all assigned neighbourhoods $U \in \mathcal{U}$ are homeomorphic to open balls in \mathbb{R}^2 . That was, of course, a remarkable contribution to the clarification of what is essential for an axiomatic characterization of manifolds. Weyl left, however, a gap, which was not surprising for the time. He dropped Hilbert’s axiom (5) to achieve stronger localization than his former teacher; but he did not realize that separability of points by neighbourhoods was thus lost. So it was left to Hausdorff, the more acute thinker with respect to logical clarification of concepts, to pinpoint the necessity of such an additional postulate in his axiomatization of topological spaces (Hausdorff, 1914, p. 213).⁶⁵

⁶⁴ Compare [Scholz, 1996; Aull and Lowen, 1997].

⁶⁵ Weyl was relatively slow to accept the necessity of a separability axiom for manifolds. He did *not* supplement or change his axiomatics of two-dimensional manifold in the second edition of (Weyl, 1913) in 1923 and did so only for the third edition in 1955. In the middle of the 1920-s he had accepted the importance of this Hausdorffian specification (Weyl, 1925/1988, p. 3). I owe R. Remmert the hint at the this gap and the relatively late correction in Weyl’s approach.

At the end of his book on the foundation of analysis *Das Kontinuum* Weyl experimented with a modified axiomatization of the concept of two-dimensional manifolds from a constructive perspective. Now he worked with a restricted real continuum, the *Weylian reals* \mathcal{W} , constructed by only those Dedekind cuts in \mathbb{Q} that are definable in a semiformalized arithmetical language (essentially using first order predicate logic and recursive definitions over \mathbb{N}) (Weyl, 1918, pp. 80ff.).⁶⁶ He postulated a “somehow” constructively given countable base of nested neighbourhoods $U_{p,n}$ ($n \in \mathbb{N}$) with $U_{p,n+1} \subset U_{p,n}$ for a countable “dense” net of points $p \in X \subset F$, each $U_{p,n}$ bijectively bicontinuous with an open disk in the Weylian number plane \mathcal{W}^2 . But for philosophical reasons he was discontented with his new approach just from the beginning.⁶⁷ A little later he turned towards Brouwer’s approach in the foundation of mathematics, even if only for a while, and essentially became an adherent of a constructive (combinatorial) approach to manifolds.

5.2. The rise of the combinatorial and piecewise linear approach

Other mathematicians had already started to pursue such another, more constructive approach to a modern formulation of the manifold concept, following Poincaré’s decomposition of manifolds into geometric cell complexes (“polyèdres”). Already Dehn and Heegard in their article on *Analysis Situs* for the *Encyclopädie der Mathematischen Wissenschaften* emphasized the combinatorial construction of manifolds, which was intended as a *definition*, not as a reconstruction of an object that had already been given in another way. In consequence they explicitly introduced the idea that morphisms of these objects should be defined by combinatorial equivalence⁶⁸ rather than by (bicontinuous) homeomorphism. Such an approach was also chosen by H. Tietze in his *Habilitationschrift* in which he studied manifolds as n -dimensional cell complexes up to combinatorial equivalence. To specify manifolds among more general cell complexes he postulated that the star of each m -dimensional cell C^m , i.e. the union of all higher dimensional cells that intersect the boundary of C^m be *simply connected*, by which he understood that it is combinatorially equivalent to a sphere S^{n-m-1} (Tietze, 1908, p. 24).⁶⁹ He left open, however, how such an equivalence could be identified.

As a great advantage of this approach Tietze observed that it would lead to a foundation of analysis *situs* independent of the consideration of infinite sets and their inherent logical difficulties and methodological subtleties (1908, p. 2).⁷⁰ As a contribution to such subtleties (at least from the point of view of Poincaré) he discussed cell decompositions of the same manifold M with infinitely many components of the intersection of cells. Thus he showed that Poincaré’s conviction that each two (finite) cell decompositions have a common subdivision was too rash to be accepted. He admitted that a proof of the existence of

⁶⁶ [Feferman, 1988; Coleman and Korte, 1998].

⁶⁷ Compare [Scholz, 1998].

⁶⁸ Dehn and Heegard used Möbius’ terminology of “elementary relationship (Elementarverwandtschaften)” (Dehn and Heegard, 1907, pp. 159f.).

⁶⁹ Tietze used the terminology of “homeomorphism” to S^{n-m-1} , but made it clear that he understood in this context “homeomorphism” in the sense of combinatorial equivalence (Tietze, 1908, p. 13).

⁷⁰ At the time of publication of Tietze’s studies the principle of choice, which had been used by Zermelo (1904) a little earlier and explicitly introduced as an axiom of set theory the same year (Zermelo, 1908a, 1908b), led to intense debate and controversial reactions among mathematicians in France and Germany. Cf. [Moore, 1978, 1982].

such a common subdivision “might be relatively simple in the case of two dimensions”, but that it “waits for a deeper investigation (harrt einer eingehenderen Erledigung) in the general case of higher dimensions (Tietze, 1908, p. 14). So he openly posed the question, whether two homeomorphic manifolds M and M' are in fact combinatorially equivalent, as an important problem of the theory. In the 1920-s H. Kneser emphasized the methodologically central role of this conjecture even more strongly and gave it the famous name of *Hauptvermutung* for the combinatorial theory of manifolds (Kneser, 1926, p. 6).

In another publication of the same year E. Steinitz attempted an axiomatic foundation of combinatorial topology by a set of postulates for the incidence structure of abstract finite cell complexes constituted by a finite set of “elements” a graded by “dimension” $\dim a = [a]$ ($0 \leq [a] \leq n$) and with prescribed incidence relations. After the introduction of six axioms to regulate the concept of an abstract combinatorial polyhedron Steinitz added three more to specify what he considered as “combinatorial manifolds”. Strangely enough he only postulated connectedness (axiom 7), existence of bounding cells for cells of intermediate dimension ($2 \leq [a] \leq n - 2$) and connectness of the boundary set (axiom 8) and existence of incident cells $[c]$ of each intermediate dimension $[a] < [c] < [b]$ to each two incident cells a and b of dimensional difference at least 3 (axiom 9) (Steinitz, 1908, pp. 37f.).⁷¹ Of course, Steinitz also introduced a combinatorial concept of equivalence for his abstract cell complexes (and “manifolds”); but although his axiomatization broke new ground for an abstract approach towards combinatorial topology in general, his characterization of manifolds was much too weak to be accepted or of broader influence for future research. So it was in fact Brouwer’s highly influential introduction of a “mized approach” of combinatorial and continuity methods, in which manifolds were defined by simplicial methods, that marked the next remarkable leap for a constructive underpinning of the manifold concept. It also pointed out in which direction one had to go if manifolds should be selected among the more general objects of abstract combinatorial complexes.⁷²

Brouwer seems to have detected the importance of simplicial decomposition of manifolds, of simplicial approximation, and of mapping degree for the investigation of long standing problems in the topology of manifolds early in 1910.⁷³ He introduced his new tools of simplicial approximations and the mapping degree of continuous maps between manifolds in his famous publication (Brouwer, 1911b). There he defined manifolds in a manner adapted to his context of the simplicial methodology. He explained an n -dimensional manifold M to be a (possibly) infinite⁷⁴ *geometric simplicial complex* of dimension n such that:

- (1) two intersecting n -simplexes share a p -dimensional face ($1 \leq p \leq n - 1$) and with it all lower dimensional faces of the latter,
- (2) for each vertex the collection of incident simplexes is homeomorphic to an n -ball (Brouwer, 1911b, p. 97).

⁷¹ Compare [Volkert, 1994, pp. 173ff.].

⁷² For an outline of how Brouwer’s intuitionism and his topological constructivism went in hand see [Koetsier and van Mill, 1997].

⁷³ See Freudenthal’s evaluation of an unpublished notebook of Brouwer in (Brouwer, 1976, pp. 422–425); compare also [Johnson, 1987, pp. 81ff.].

⁷⁴ In the case of a finite simplicial decomposition he called the manifold “closed” (in our terminology compact), in the infinite case “open”.

If all n -simplexes of an n -dimensional manifold M are represented by $n + 1$ homogeneous coordinates of a standard simplex in the $(n + 1)$ -dimensional “number space”,⁷⁵ such that lower dimensional simplexes carry identical coordinates from each n -simplex to which they belong, Brouwer called the manifold “measured”, in more recent terminology M carries a *piecewise linear* (abbreviated PL-) structure. By a recursion procedure over the dimension he showed that a manifold in his sense can always be “measured” (given a PL-structure). That allowed him to characterize orientability and orientation of his PL-manifolds, barycentric subdivisions, simplicial approximation of continuous maps, and the mapping degree of continuous maps between manifolds. That served as the basis for his investigation of the index of vectorfields on spheres, his fixedpoint theorem, the proof of the invariance of dimension, etc.⁷⁶ Thus he introduced a new approach for a constructive characterization of manifolds besides the less standardized representations as geometric cell complexes in the sense of Poincaré, Tietze et al.

Brouwer’s approach to manifolds combined a constructive representation of the global structure by “measured” simplicial complexes with a criterion of local simplicity, which still referred to pointset topological properties of “numberspaces” and did not even attempt to transform the latter to combinatorial criteria. In this respect Tietze (and Steinitz) had been more consequential in their attempt to avoid the fallacies of pointset topology. They had successors to elaborate more in detail, what Tietze had left open in his all-inclusive characterization of “simple connectedness” of neighbourhoods of k -cells. In the early 1920-s Veblen and Weyl pushed this characterization a step further, although they were not completely successful in their search for a convincing and operative characterization.

5.3. Manifolds in the methodological “battles” of the 1920-s

Veblen followed in his *Analysis Situs* (1922) the combinatorial approach to manifolds and complemented it by ingredients from Brouwer’s simplicial constructs. Like Tietze he explicitly tried to avoid pointset topological considerations as far as possible.⁷⁷ He modified Tietze’s combinatorial definition by a rather pragmatic reduction of the combinatorial problem to characterize “simply connected” stars of k -cells in an n -dimensional complex.⁷⁸ After giving three procedures to build an n -complex combinatorially equivalent to an n -cell,⁷⁹

⁷⁵ I deliberately use Brouwer’s original terminology and do not write \mathbb{R}^n , as Brouwer’s terminology leaves the interpretation of the number continuum open. It can be interpreted by classical real numbers, Brouwer’s intuitionistic real continuum, or even (later) by Weylian reals \mathcal{W} of 1918.

⁷⁶ Cf. [Johnson, 1987; Koetsier and van Mill, 1997].

⁷⁷ “. . . we leave out of consideration all the work that has been done on the point-set problems of analysis situs and on its foundation in terms of axioms or definitions other than those actually used in the text.” (Veblen, 1922, p. vii). In consequence Hausdorff did not start to read Veblen’s and other mathematicians’ work on combinatorial topology before the late 1920-s when Alexandroff’s approach to homology via “nerves” of open coverings allowed an algebraization of homology which was directly applicable to topological spaces independent of a combinatorial structure.

⁷⁸ In addition Veblen choose to give the explanation of a “neighbourhood” of a k -cell $a^{(k)}$ in an n -dimensional complex C_n a surprising shift towards pointset theoretic criteria. He characterized such a “neighbourhood” as any set S of nonsingular cells of C_n such that all point sets $M \subset C_n$ with a limit point on $a^{(k)}$ have points in S (Veblen, 1922, Chapter III.4).

⁷⁹ The simplest of these three procedures was of course the following: Two nonintersection n -cells $a_1^{(n)}$, $a_2^{(n)}$ incident with exactly one $(n - 1)$ -cell $a^{(n-1)}$ constitute an n -cell.

Veblen defined an n -dimensional manifold to be a closed (finite) regular cell complex of dimension n , in which each k -cell has a “simply connected” star, where “simple connectedness” was defined by a combination of the three construction procedures of k -cells given before.⁸⁰

H. Weyl was at that time highly impressed by Brouwer’s ideas of “free choice sequences” to characterize continuum ideas mathematically without reference to the conceptual framework of transfinite sets. So he experimented at the end of his polemical article on the *new foundational crisis of mathematics* with a characterization of point localization in a two-dimensional continuum by “free choice sequences” of nested stars in an infinite series of barycentric subdivisions of a two-dimensional Brouwerian manifold (Weyl, 1921, p. 177f.). He tried to come to a genetic definition of points in a two-dimensional combinatorial continuum and rejected the idea that “points” might be presupposed as ideal “atomistic” local determinations in advance.⁸¹ In that respect the “purely” combinatorial approach to manifolds appeared to him of high importance for the foundations of mathematics, the more so as he could not be sure that Brouwer’s intuitionistic continuum (which in the early 1920-s was not yet well elaborated in technical details) and his own ideas on an “infinitesimal continuum” would conceptually coincide after sufficient symbolical elaboration. In any case, it would be logically preferable to free the combinatorial approach to continuum concepts from the direct link to number concepts, which was presupposed in Brouwer’s “mixed” approach to manifolds.

Thus Weyl took the opportunity of his visit to Madrid and Barcelona in early 1922 not only to elaborate his ideas on the “analysis of the space problem” from the new viewpoint of his infinitesimal geometric approach but also to give an exposition of his view of combinatorial topology for the *Revista Matemática Hispano-Americana* (Weyl, 1923, 1924). He developed his own approach to a characterization of a combinatorial n -sphere (a “Zyklus” as Weyl said) by two groups of axioms. As there was neither a semantically complete axiomatic characterization of combinatorial spheres, nor a complete set of construction procedures for the latter in sight, Weyl proposed for the time being a provisional axiomatic characterization of structural properties of combinatorial spheres “from above” in a first group of axioms, and in addition a second group of axioms, which gave a collection of genetic procedures for the construction of combinatorial k -spheres “from below”. He hoped for a step by step completion of the axiom system in future research, which in the end might lead to a coextensive characterization of combinatorial k -spheres by any of the (extended) two groups of axioms, and thus of manifolds.⁸²

The first group of axioms for a Weylian combinatorial n -sphere Z^n were the postulates that Z^n be connected (axiom 1), that to each k -cell $a^{(k)}$ in Z^n ($0 \leq k < n$) the collection of all higher dimensional cells $b^{(j)}$ bounding directly or indirectly ($k < j \leq n$, $a^{(k)} \subset \bar{b}^{(j)}$) carries the combinatorial structure of a Weylian combinatorial sphere Z^{n-k-1} (axiom 2),⁸³ that it be orientable (axiom 3), and homologically trivial in dimensions less than n (axiom 4).⁸⁴

⁸⁰ (Veblen, 1922, Chapter III.24).

⁸¹ For more details on Weyl’s philosophical motivation and the context of his rejection of transfinite set theory as a background in which to model “continuum” ideas, compare [Scholz, 1998].

⁸² Cf. (Weyl, 1924, pp. 416f., 419) and also (Weyl, 1925/1988, p. 10).

⁸³ Weyl called this (modified) Tietzean property of a cell complex to be “unbranched”.

⁸⁴ Weyl called this a “plain (schlicht-de una hoja)” complex (Weyl, 1923, p. 403).

For the “genetic” characterization of combinatorial n -spheres Weyl characterized the 0-sphere as two points (axiom 0), generation of n -spheres from n -spheres by subdivision of a rather general kind⁸⁵ up and down (axioms A, B) and two constructions of higher dimensional spheres from lower dimensional (axioms C, D). Weyl proved several results for combinatorial manifolds, in particular Poincaré duality for closed orientable manifolds. But his approach was probably too involved in foundational considerations and technically too sophisticated to be taken up as a convincing strategy for the elaboration of a more broadly accepted genetic concept of manifold, which stood up to the standards of modern mathematics.⁸⁶ So the research strategy proposed by Weyl was not taken up by other mathematicians, but at the best selectively adapted to other methodological views.

About the middle of the 1920-s all ingredients for a satisfying formulation of the manifold concept, taking up the knowhow on axiomatization and on genetic characterizations of manifolds were at hand. There is no point in repeating here the interesting history of the preparation and elaboration of the general concept of topological space.⁸⁷ After Hausdorff, Fréchet, and Riesz opened this new field of investigation, it found particularly active supporters in the newly rising mathematical groups in Poland, the Soviet Union, and the United States, and it also left its imprint on the modern reframing of mathematics in Germany and Austria. The first attempt for balance between the different approaches to the manifold concept was given by the young Hellmuth Kneser, who had written his dissertation with Hilbert in 1921 and got a professorship in Greifswald in 1925, in an article for the *Jahresbericht der DMV* (Kneser, 1926).

Kneser discussed both approaches, an axiomatic one based on Hausdorff’s set theoretic foundations for topology, and a combinatorial one referring to, but deviating from Weyl’s approach. For the axiomatic characterization of manifolds he limited himself to the topological case, without any discussion of differentiable structures. He thus characterized a *topological manifold* M by Hausdorff’s axioms for a neighbourhood basis (of a Hausdorff space) including the second countability axiom for a neighbourhood base of all points in M (thus of all open sets in M) and added just one postulate: Each point $p \in M$ has a neighbourhood which is topologically equivalent to an open ball in the “ n -dimensional numberspace”, by which he obviously referred to the \mathbb{R}^n (Kneser, 1926, pp. 1–3). A “closed” (in our terminology compact) manifold was characterized by the Heine–Borel criterion for open coverings of M .

After the introduction of a combinatorial decomposition of a “closed” manifold as a finite cell complex Kneser introduced the *Hauptvermutung* as fundamental for the combinatorial theory of manifolds and proposed a characterization of a *combinatorial manifold* by the simultaneously inductive definition of the concepts of n -dimensional cell complex C^n , n -dimensional combinatorial sphere S^n , n -dimensional cell E^n , the boundary of E^n , and internal transformations (allowed subdivisions of cells of S^n -s). He first defined by

⁸⁵ For subdivision of an n -cell E^n Weyl used any combinatorial n -sphere Z^n punctuated it and substituted it for the E^n .

⁸⁶ H. Kneser (1926) referred to Weyl, and gave a little later van Kampen the hint that the latter’s cell complexes satisfy Weyl’s axioms (van Kampen, 1929, p. 3). On the other hand Kneser criticized Weyl’s axioms as too complicated, as the consistency and completeness of the axioms were left open (1926, pp. 12f.). The former aspect (consistency) had in fact been discussed by Weyl (1924, pp. 416ff.), while completeness had been marked as a severe problem by the latter (1924, p. 419).

⁸⁷ See diverse articles in the recent *Handbook* [Aull and Lowen, 1997] and the announced next volume(s) with several contributions on the history of set theoretic (“general”) topology.

induction over dimension n , what might be called a regular cell complex⁸⁸ C^n , where essentially each n -cell Z^n in C^n has a combinatorial sphere S^{n-1} as boundary, and the $(n-1)$ -subcomplex is also regular. Moreover for all k -cells Z^k in C^n ($1 \leq k \leq n$) internal transformations by elementary cell subdivisions (or the inverse operation) are defined.⁸⁹ Starting from standard combinatorial n -spheres⁹⁰ S_N^n Kneser allowed all those combinatorial schemes of spheres, which can be constructed by internal transformations in his sense. Thus he was proud to achieve a simpler characterization of combinatorial manifolds M^n than Weyl, by the condition that the neighbourhood complex of each 0-cell Z^0 has an S^{n-1} as boundary.⁹¹ But his approach was not only built on the unproven (and unprovable as we know) Hauptvermutung, but did even not allow the proof of Poincaré duality for orientable manifolds by combinatorial means. So it was in the end doubtful whether his approach had a real advantage in comparison with Weyl's, although he had achieved a much simpler framework of postulates.

In the late 1920-s several mathematicians in different international groups, relatively independent from each other, turned towards a more pragmatic approach with respect to combinatorial manifolds. They turned the question upside down⁹² and looked for combinatorially accessible conditions that an orientable cell complex satisfies "Poincaré" duality. J.W. Alexander, L.S. Pontrjagin,⁹³ L.F. Vietoris (1928), and E.R. van Kampen in his Leiden dissertation (1929) chose similar strategies to get rid of the unanswerable question under which conditions a combinatorial complex is a "real" (i.e. topological) manifold. The essential common point of their approaches was the idea of weakening the sphere condition for the boundaries of neighbourhood complexes from combinatorial to purely homological ones. In this sense Vietoris took up Brouwer's constructive definition of a manifold and modified it by a homology criterion for the local simplicity property, defined inductively over dimension. More precisely, he defined an *h-manifold* as a simplicial complex M in which the star of each vertex e_0 is bounded by an "*h-sphere*". An $(n-1)$ -dimensional *h-sphere*, on the other hand, is defined as an orientable *h-manifold* of dimension $n-1$ with the same Betti numbers as a sphere: $p_0 = p_{n-1} = 1$, $p_i = 0$ for $1 < i < n-1$; with the inductive definition anchored in the obvious stipulation that a 0-dimensional *h-sphere* is a pair of points (Vietoris, 1928, p. 170). That allowed him to establish Poincaré duality for orientable closed *h-manifolds* by the construction of dual complexes and the use of Poincaré's argument. In fact, in the introduction of his paper he stated frankly that his proposal of a modified concept of *h-manifolds* arose from a proof analysis as a result of which he did not try to fill the gap in the original argumentation, but preferred to adapt the conceptual frame to Poincaré's original proof structure.⁹⁴

⁸⁸ Kneser used the terminology "cell building (Zellgebäude)".

⁸⁹ More precisely the boundary of each Z^k which is a S^{k-1} is divided into two standard $(k-1)$ -cells with common boundary S^{k-2} . Then the substitution of Z^k by two k -cells Z_1^k, Z_2^k , and a $(k-1)$ -cell Z^{k-1} which are bounded by the subdivided parts of the S^{k-1} and inherit the boundary relations of the large cell Z^k is an *internal transformation* in the sense of Kneser (1926, p. 8).

⁹⁰ S_N^n is defined by two cells in each dimension $0, \dots, n$, each of which is bounded by all cells of less dimension.

⁹¹ Kneser claimed that by use of the internal transformations the same holds for "all the other points".

⁹² From the point of view of the manifold concept one should perhaps say that they turned the question "downside up".

⁹³ Alexander and Pontrjagin's in unpublished notes, as was reported by van der Waerden (1930, p. 125); compare also [Dieudonné, 1989, p. 50].

⁹⁴ "We shall not fill this gap (of the original proof referring to manifolds, ES) but define a manifold concept for which we can fill it, while the remaining proof of Poincaré can be transferred without change." (Vietoris, 1928,

Lefschetz generalized this approach in terms of relative homology with respect to a subcomplex, thus documenting that the combinatorial strategy to work out the manifold concept was deeply influenced and even transformed by the advent of algebraic topology (Lefschetz, 1920, pp. 119ff.).

Van Kampen followed an approach closer to Veblen's and Weyl's recursive definition of manifolds. During his doctoral research he was in contact with B.L. van der Waerden and informed by the latter about the different strategies for coming to a formally satisfying definition of the concept. Van Kampen chose to add to the basic structure of a Brouwerian simplicial complex the structure of what he called a star-complex, where the concept of *star* and *star complex* had a common recursive definition.⁹⁵ Equality of star-complexes STC^n and $STC^{n'}$ was defined by him as a combinatorial equivalence of the underlying simplicial complexes, which leads to a bijection of the stars. Thus the incidence structure of the stars (of all dimensions) gives complete information about the structure of a star-complex,⁹⁶ and allowed him to define a dual star-complex STC^{n*} to a given star-complex STC^n with the same underlying simplicial complex and dualization $k' = n - k$ of the dimensions k, k' of dual stars.⁹⁷ Then the incidence matrices of a star-complex and its dual arise from interchanging order of the columns and transposition and behave like Poincaré's incidence matrices in the proof of Poincaré duality.

Van Kampen had thus won a recursively defined normalization of simplicial complexes to which he added a postulate with a dual combinatorial criterion of local simplicity for defining a *combinatorial manifold*: (1) Each k -star is homologically trivial in dimensions $1 < j < k$; and (1') the same holds for the stars of the dual star-complex (van Kampen, 1929, p. 13). The approach was chosen to derive different duality theorems (Poincaré-, Alexander-, etc.) in a purely combinatorial and thus finite manner. Moreover the combinatorial manifolds satisfy Weyl's axioms, as van Kampen remarked with reference to H. Kneser, but without any discussion of Weyl's original goal to sharpen increasingly the combinatorial postulates until they are coextensive with an axiomatic characterization of continuous manifolds.⁹⁸

The next year B.L. van der Waerden gave a talk at the annual meeting of the *Deutsche Mathematiker-Vereinigung*, in which he presented and discussed the different proposals for the definition of a topological manifold on what he called the "battlefield of different methods" in combinatorial topology (van der Waerden, 1930, p. 121). He counted 5 different possibilities, an axiomatic one (Kneser, 1926), two purely combinatorial ones, of which he presented one as methodologically unsatisfying (Dehn and Heegard, 1907, Tietze, 1908) and the other, homologically oriented one, as more sophisticated (Vietoris, 1928; van Kampen, 1929), and "two" mixed approaches (Poincaré, 1899, 1900; Brouwer, 1911b). Van der

p. 165). Here we have, to put it in Lakatos' terminology, a beautiful case of a completely conscious concept modification generated by proof analysis.

⁹⁵ A star of dimension 0 is a point; a 0-dimensional star-complex is a finite set of stars. An n -dimensional star is a (simplicial) projection of an $(n - 1)$ -dimensional star-complex from a point (the *centre* of the star). An n -dimensional star-complex STC^n is produced from an $(n - 1)$ -dimensional star-complex STC^{n-1} by adding n -stars, generated by projection of star-subcomplexes of STC^{n-1} , such that each star from the latter is part of the border of at least one of the n -stars (van Kampen, 1929, pp. 6f.).

⁹⁶ (van Kampen, 1929, Theorem 2a,b).

⁹⁷ A k -dimensional star a_i^k of STC^n is dualized by collecting all $(n - k)$ -dimensional stars b_j^{n-k} which meet the centre of a_i^k , but no other vertex.

⁹⁸ (van Kampen, 1929, p. 3).

Waerden discussed the relative merits and disadvantages of all these approaches.⁹⁹ The phase of open exploration for the topological manifold concept had more or less come to a conclusion; the axiomatic characterization and a constructive (purely combinatorial) one were the outcome of differing methodological approaches, Brouwer’s “mixed” approach gave the most promising bridge, and Weyl’s original intentions were close to forgotten.¹⁰⁰

5.4. Finally the “modern” axiomatic concept

There was, of course, still another line of research, more closely linked to differential geometry, where manifolds played an essential role, and purely topological aspects (independently of whether continuous, combinatorial, or homological ones) did not suffice and still needed elaboration. In North America Oswald Veblen and his students formed an active center in both fields of topology and modern geometry. Veblen and his student J.H.C. Whitehead, coming from (and going back to) Oxford, brought the axiomatization of the manifold concept to a stage which stood up to the standards of modern mathematics in the sense of the 20th century (Veblen and Whitehead, 1931, 1932). Veblen was an admirer of the Göttingen tradition of mathematics, in particular, F. Klein and D. Hilbert, and cooperated closely with H. Weyl, the broadest representative of his own generation from the Klein and Hilbert tradition. Veblen and J.H.C. Whitehead combined a view of the central importance of structure groups for geometry (generalizing the Erlanger program) with Hilbert’s embryonic characterization of manifolds by coordinate systems; and they took care that the topologization of the underlying set would satisfy Hausdorff’s axioms for a topological space.

They characterized the *structure* of a manifold by the specification of a regular groupoid G (“pseudogroup”) of transformations of open sets (“regions”) in \mathbb{R}^n , allowing as main examples C^1 -transformations of open sets ($i = 0, \dots, \infty$, or $i = \omega$). The n -dimensional manifold of structure G in the sense of Veblen and Whitehead consists in a set M and a system of *admissible coordinate systems* $\varphi: U \rightarrow V$ with bijective maps φ onto regions $V \subset \mathbb{R}^n$, defined for $U \in \mathcal{U} \subset \mathcal{P}(M)$, such that three groups of axioms hold:

(A) *Basic axioms for admissible coordinate systems:*

Changes of coordinates are given by maps from the structure groupoid G and each coordinate map may be changed by a transformation from G (axioms A_1, A_2). Moreover, to each coordinate map $\varphi: U \rightarrow V$ a restriction to $U' \subset U$ such that $\varphi(U') = V'$ is an n -cell V' in \mathbb{R}^n is also an admissible coordinate system (A_3). U' is called an n -cell in the manifold.

(B) *Union of compatible coordinate systems:*

If for a collection of admissible coordinate systems $\varphi: U_i \rightarrow V_i$ ($i \in I$), with n -cells as coordinate images V_i , the coordinate maps coincide on overlaps ($U_i \cap U_j \neq \emptyset$), then the “union” of coordinate systems defined in the obvious way, $\varphi: \bigcup_i U_i \rightarrow \bigcup_i V_i$, is

⁹⁹ The references between the 5 approaches to authors were not all made explicit by van der Waerden but presented on a purely methodological level.

¹⁰⁰ Weyl’s contribution appeared in van der Waerden’s bibliography, but was not discussed by him. He thus indirectly took part in the methodological “battle” of combinatorial topology, although he probably did not realize it.

also admissible (axiom B_1). Each admissible coordinate system can be represented as such a union (B_2).¹⁰¹

(C) *Topological axioms:*

For intersecting n -cells U, U' in M with $p \in U \cap U'$ there is an n -cell $U'' \subset U \cap U'$ containing p (axiom C_1). For each two different points $p, q \in M$ there exist nonintersecting coordinate neighbourhoods U_p, U_q of p and q , respectively (C_2).

Finally, M contains at least two different points (C_3).

Taking n -cells in M , containing p , as neighbourhoods of p the axioms of Veblen and Whitehead give a structure of a Hausdorff space on M (without second countability axiom) (Veblen and Whitehead, 1931, p. 95; 1932, p. 79).

Whitehead and Veblen presented their axiomatic characterization of manifolds of class G first in a research article in the *Annals of Mathematics* (Veblen and Whitehead, 1931) and in the final form in their tract on the *Foundations of Differential Geometry* (Veblen and Whitehead, 1932). Their book contributed effectively to a conceptual standardization of modern differential geometry, including not only the basic concepts of continuous and differentiable manifolds of different classes, but also the “modern” reconstruction of the differentials $dx = (dx_1, \dots, dx_n)$ as objects in tangent spaces to M .¹⁰² Basic concepts like Riemannian metric, affine connection, holonomy group, covering manifolds, etc. followed in a formal and symbolic precision that even from the strict logical standards of the 1930-s there remained no doubt about the wellfoundedness of differential geometry in manifolds. Moreover they made the whole subject conceptually accessible to anybody acquainted with the language and symbolic practices of modern mathematics.

5.5. And first successes in unification

The clear definition and mutual delimitation of continuous, differentiable and analytic structure of manifolds by Whitehead and Veblen improved the framework for a more detailed study of the basic questions of triangulation, Hauptvermutung and thus the questions which were at stake with the competing strategies of a genetic/constructive characterization of manifolds versus an axiomatic one. They had been posed at first for topological manifolds, but could as well be fruitfully transferred to the differentiable case.

Already at the turn of the thirties, i.e. before the Veblen and Whitehead axioms had been formulated, first positive results on the connection between the two large strategies had been achieved. In 1925 T. Radó had shown that two-dimensional manifolds can be triangulated and thus that in this respect Tietze had been right. During the following decade the higher dimensional case could only be dealt with under structurally specifying conditions. Several authors contributed to the proof that a real analytical manifold admits triangulation: Van der Waerden (1929) clarified the triangulability for algebraic manifolds, Lefschetz (1920, Chapter VIII) sketched the outline for a general proof in the case of a general analytic manifolds, and Koopman and Brown (1932) elaborated a complete proof. Only a few years later S.S. Cairns, a former student of M. Morse, proved the existence of triangulations for n -dimensional differentiable manifolds (Cairns, 1934). In 1940 J.H.C. Whitehead considered and showed the existence of differentiable triangulations of a C^1 -manifold of

¹⁰¹ Neither here nor elsewhere did Veblen and Whitehead postulate a countability restriction for the coordinate neighbourhoods.

¹⁰² They still used the pre-Bourbakian terminology of “contravariant” vector for the objects in the tangent space.

any dimension and proved that in this structure even the (differentiable) Hauptvermutung is true.¹⁰³ Thus the combinatorial and the axiomatic approach had turned out by 1940 to be complementary aspects of completely coextensive characterizations for *differentiable* manifolds.

So far Poincaré’s intuition had been vindicated and put on a solid logical basis, although the elaboration of “purely” (continuity) topological structure had shifted the question to a deeper conceptual level than Poincaré ever would have considered.¹⁰⁴ And even the topological case seemed at first rather promising, at least in low dimensions. Radó’s success for dimension 2 was extended in the early 1950-s, when E.E. Moise proved that each 3-dimensional continuous manifold admits a triangulation (Moise, 1952). At the turn of the 1950-s one might thus have hoped that the different specifications of the manifold concept had led to difficult and challenging technical problems for modern mathematics, but that they could perhaps be solved positively by increasingly sophisticated methods and an interplay between the different structural level and methods. Why should they not lead to a unified frame for the topology and geometry of manifolds in a rather straightforward manner?

6. Outlook on more recent developments

6.1. Growing diversity. . .

Even the conceptual unity one might have hoped for in the early 1950-s was, however, not at all a narrow one. Already Riemann had indicated the possibility of investigating manifolds from different methodological views and had considered this differentiation as an important feature for adapting the general concept to diverse scientific contexts. Such a differentiation had developed on a technically much more refined level during the first half of the 20th century in a broader range. Besides the distinction between the combinatorial or PL- and axiomatic approaches to the topological manifold concept and its differentiation according to smoothness levels (C^i , $0 \leq i \leq \infty$ or $i = \omega$), other contexts had given reasons for developing the concepts of a complex analytic manifold and of algebraic birational variety. These, as well as the diverse differential geometric structural specifications on differentiable manifolds, would have to be considered for a broader picture of the growing diversity of manifolds in our century, but remain outside the range of this article.

To keep closer to the core of our subject, we have to face the surprising diversity in the topological and differential structures on manifolds of dimension $n \geq 4$, which became apparent by and by starting in the late 1950-s. After J. Milnor detected nonstandard differentiable structures on the 7-sphere (Milnor, 1956), an increasing number of unexpected insights into the differentiable structure of higher dimensional manifolds came to the fore. Among them were E. Brieskorn’s and others’ study of exotic spheres, which arose relatively “naturally” in investigations of singularities of algebraic geometry, and in the 1960-s M.H. Freedman’s and S.K. Donaldson’s broad investigations of differentiable structures on 4-manifolds. During the 1980-s the tremendous range of effects, from a number of unexpected differentiable structures on supposedly well known manifolds, like higher dimensional spheres and the \mathbb{R}^4 , to the fact that certain topological 4-manifolds do not admit

¹⁰³ (Whitehead, 1940, Theorem 8, p. 822).

¹⁰⁴ Poincaré considered his manifolds always as differentiable, in times even as analytic, which he defined by an approximation argument of analytic maps by differentiable ones.

a differentiable structure at all, became known. They would have given sufficient reason for a Poincaré to deplore again, and now on another much more sophisticated level, the turn of mathematics towards an “artificiality which alienates the whole world”, as he had proclaimed in his talk to the second International Congress of Mathematicians with respect to the rise of modern mathematics (Poincaré, 1902c). He was particularly struck by the results of the logical analysis of continuous nondifferentiable functions.¹⁰⁵ The preparation of such unexpected symbolical phenomena was nevertheless an important part of the achievements of the high phase of modern mathematics and characteristic for its spirit. They are discussed more in detail and with much more expertise in other contributions to this volume.

Similar evaluations might be drawn on the fate of the triangulation problem and the Hauptvermutung for topological manifolds of dimension $n \geq 5$. J. Milnor’s early example of manifolds in dimension 8, with different combinatorial structures (Milnor, 1961) paid tribute to but finished the hope for a too simply conceived positive end of the program outlined in the first third of the century. The work by R.C. Kirby and L.C. Siebenmann (1969) with the characterization of exact obstruction criteria, given by cohomology classes of the manifold in question, allowed their successors to determine manifolds for all dimensions $n \geq 5$, in which the Hauptvermutung does not hold, and to characterize topological manifolds without any PL-structure.

Thus from the late 1960-s onward the hope for a conceptually unified framework for all of modern mathematics has undergone a deep transformation,¹⁰⁶ forced upon the research community by the growing complexity of material, methods and results.¹⁰⁷ There is a growing and perhaps ever increasing trend towards diversification, and differentiation even, to a certain degree, between different subbranches or aspects in the mathematics of such a relatively well delimited field as the topology and differential topology of manifolds.

6.2. ... but still a unifying perspective on mathematical practice by overarching concepts

Some observers even tend to see a loss of connection between different branches of mathematics as a whole and identify such a loss of unity, growing pluralism of methods, structures, and approaches from a specific cultural perspective as a “postmodernist” dynamics of mathematics, which has speeded up from the late 1960-s onward. No doubt, modern mathematics, and maybe with it, modern culture has reached a mature, probably even “late” stage, at least in comparison with its expansionary “high” phase from the late 19th to the middle of the 20th century.¹⁰⁸ But history has always been an open process, and Riemann’s and other persons’ vision of the cognitive strength and productivity of conceptual unification has neither lost its fruitfulness nor its convincing power.

The vision of a strictly unified and structurally predetermined symbolical universe of mathematics, which seems to have been the dream of many of the protagonists of the high

¹⁰⁵ The famous citation of the monster functions which he abhorred is in (Poincaré, 1908); H. Mehlertens describes this as Poincaré’s “antimodernist” view of mathematics [Mehlertens, 1990, Chapter 3.3].

¹⁰⁶ First glances of such an ongoing shift could probably be seen already in the late 1950-s by very sensitive observers.

¹⁰⁷ Compare also [Corry, 1996] with respect to the fate of the structural “image” in recent algebra.

¹⁰⁸ I. James has called this phase of modernity as the “classical” one in his Nice talk.

phase of mathematical modernity,¹⁰⁹ has become obsolete through the progress of mathematical work itself. But we are not at all obliged to understand mathematical concepts in a rigid, mainly technical sense; we may also conceive them as organizing centers of cognition, which act in a “dialectical” interplay between their role of cognitive orientation and the symbolical and technical specification they impart on the practice of mathematics. There is no good argument to reduce, or even to proclaim the end for, the unifying role of concepts in today’s and future mathematics and human knowledge more general. We could just as well draw the opposite conclusion and insist on an increasing importance of their unificatory role as a counterbalance to cultural and cognitive diversification. Thus the history of the manifold concept may be taken as paradigmatic for a symbolical world of increasing diversity and richness in which we live, work, and orient ourselves.

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¹⁰⁹ Such protagonists were, among others, D. Hilbert and N. Bourbaki. Deviating visions existed, like Hausdorff’s or, from a completely different philosophical background, Brouwer’s and Weyl’s, but were of a much more restricted influence in the scientific world during the phase of “high” modernity.

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