



Introduction to symplectic geometry through fixed point theorems

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Outline

- 1 Fixed Points
- 2 Poincare's Last Geometric Theorem
- 3 Arnold Conjecture

What is a fixed point?

- X is a set - for example all people in this room but more typically points of a geometric shape
- $T : X \rightarrow X$ a map - each element x of X is sent to an element $T(x)$ of X
- If $T(x) = x$, we call $x \in X$ a fixed point of T
- If X has at least two elements, then there are T 's with no fixed points

Brouwer fixed point theorem

- Take X to be a unit ball B^n including its boundary in the Euclidean space $\mathbb{R}^n$
- $B^1 = [-1, 1]$, B^2 is the unit disk, B^3 is the unit ball...
- Take $T : B \rightarrow B$ a continuous map
- Meaning if two sequences of points converge to each other, then their images also converge to each other. The map does not tear/puncture B^n .
- Theorem (Brouwer, 1911): Such a T always has a fixed point.

Proof idea

- Assume the contrary
- The rays from $T(x)$ to x provides a way to continuously deform B^n onto its boundary, while keeping the boundary fixed at all times
- For $n = 1$, endpoints stay put. On the other hand, the only way to collapse the interval onto its boundary without tearing it is to send everything to the same boundary point. This can be made rigorous easily.
- For $n \geq 2$, our intuition suggests that such a continuous deformation should puncture the ball but this is a bit harder to show.

Lefschetz fixed point theorem

- Theorem (Lefschetz, 1926): X is a compact triangulable space, $T : X \rightarrow X$ is continuous. We define

$$\Lambda_T := \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(f_* : H_i(X; \mathbb{Q}) \rightarrow H_i(X, \mathbb{Q})).$$

Then, counted with signed multiplicities, T must have at least Λ_T fixed points.

- The exact statement is unimportant. The takeaway is that for a very large class of geometric shapes, there is a topological estimate on the number of fixed points under the continuity assumption.
- Unfortunately, sometimes $\Lambda_T = 0$ and this gives us nothing.
- Example: $X = \text{annulus in } \mathbb{R}^2$, torus in $\mathbb{R}^3$, product of $n > 1$ tori and T can be continuously deformed to the identity map through continuous maps (think of flowing with a vector field)

Sun-Jupiter-Trojan System

- Consider the Sun-Jupiter-Trojan system. This is approximately in the framework of Planar Circular Restricted 3 Body Problem.
- Work in the rotating frame where the Sun and Jupiter does not move (compare with the Coriolis force). The masses of Jupiter and Sun are μ and $1 - \mu$.
- The motion of the Trojan is given by

$$(x''(t), y''(t)) = 2(y'(t), -x'(t)) + \left(\frac{\partial \Phi}{\partial x}(x(t), y(t)), \frac{\partial \Phi}{\partial y}(x(t), y(t)) \right)$$

where

$$\Phi(x, y) = \frac{x^2 + y^2}{2} + \frac{1 - \mu}{\|(x + \mu, y)\|} + \frac{\mu}{\|(x + (1 - \mu), y)\|}$$

- An example of possible motion Video

Periodic orbits of PCR3B

- Introducing new variables $p_x = x' - y$ and $p_y = y' + x$, we turn the 2nd order ODE for $(x(t), y(t))$ to a 1st order ODE for $(x(t), y(t), p_x(t), p_y(t))$
- This means that we have the effective energy function $\frac{(p_x+y)^2}{2} + \frac{(p_y-x)^2}{2} - \Phi(x, y)$. This function is preserved during the motion. It is really a Hamiltonian function for the motion in the phase space (x, y, p_x, p_y) .
- We consider the motion of the Trojan with sufficiently small negative energy level $-C < 0$
- This 3d hypersurface has three components one living over a region near the Sun, one near the Jupiter and one near infinity (suplevel set $\Phi(x, y) \geq C \gg 1$).

Hill's Region Image from Chenciner

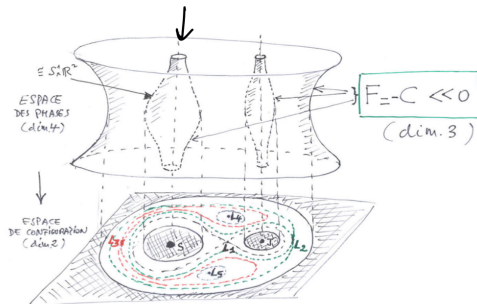


Figure 12: Hill stability.

Figure: Fig. 14 of <http://www.bourbaphy.fr/chenciner.pdf>

Transverse Annular Section

- We regularize the flow in the component near the Sun to a flow of vector field X_μ in $\mathbb{RP}^3$ using Moser regularization.
- The key to the geometry is that X_0 on $\mathbb{RP}^3$ is the Moser regularization of the Kepler problem and it's tangent to the Hopf fibers. We think of X_μ as a perturbation.
- Hill showed that for sufficiently small μ , two of the Hopf fibers persist as 1-periodic orbits.
- The Kepler problem admits an annular transversal section to the flow by taking a section of the Hopf fibration with one pole above one point and one zero above another one.
- Poincare and Birkhoff showed there is still a transversal section bounded by Hill's orbits for sufficiently small μ . Later Conley removed this assumption.

Annular Section Image from Chenciner

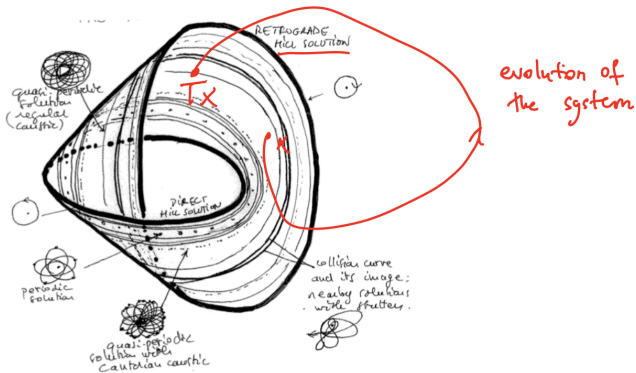


Figure: Fig. 15 of <http://www.bourbaphy.fr/chenciner.pdf>

Another perturbation: geodesics on the boundary of a convex body

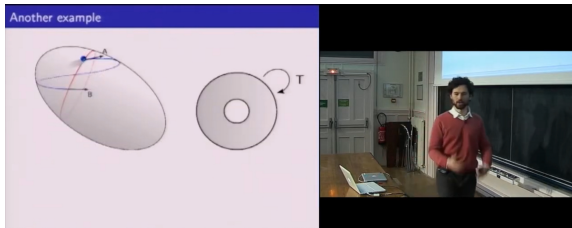


Figure: <https://youtu.be/1LQVLps4r48?si=7ITwkpiGOvrc7JVp>

Last Geometric Theorem

- Poincare showed that the return map $T : A \rightarrow A$ of X_μ is an area preserving C^1 diffeomorphism of the annulus and also it twists the boundaries in different directions (see next item).
- We can lift T to a $\mathbb{Z}$ -equivariant area preserving diffeo.

$$S : [0, 1] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}.$$

S should be strictly monotonic in both boundaries but in opposite directions.

- Poincare conjectured (in 1912, the year he died) that such a T must have at least 2 fixed points. Removing either the twist or the area preserving assumptions destroy this property.
- If correct, he would find infinitely many periodic orbits in this energy level using iterates of T – and a trick.
- Birkhoff proved that there is one fixed point in 1913, and two in 1926.

The doubling trick and center of mass and area preserving diffeomorphisms of T^2

- Consider the torus T^2 as $\mathbb{R}^2/\mathbb{Z}^2$ with standard area form $dp \wedge dq$
- An area preserving diffeomorphism of T^2 is said to preserve the center of mass, if the COM of a square fundamental domain in $\mathbb{R}^2$ for a lift diffeomorphism moves by an integral vector.
- Given $T : A \rightarrow A$ as above, by inserting a "constant movement" in between S and its reflection we can obtain an area and COM preserving diffeomorphism of T^2 .
- If we can prove that such a diffeomorphism of T^2 has at least 3 fixed points, PLGT follows!

Hamiltonian diffeomorphisms of symplectic manifolds

- Such diffeomorphisms of T^2 are special cases of Hamiltonian diffeomorphisms of symplectic manifolds
- For T^2 , area and COM preserving is equivalent to being the time 1 map of the flow of a time-dependent vector field X_t such that the signed area traced by any loop in T^2 is zero for all times.
- For a compact smooth manifold M with a differential 2-form ω we can make sense of this condition.
- If ω is closed and non-degenerate, we can equivalently require that each X_t is the Hamiltonian vector field of a function H_t :

$$\omega(X_t, \cdot) = dH_t$$

- In Darboux charts the flow satisfies Hamilton's equations from classical mechanics

Arnold: C^1 -small Hamiltonian diffeomorphisms, critical points of functions

- Arnold realized that if $\phi : (M, \omega) \rightarrow (M, \omega)$ is a Hamiltonian diffeomorphism that is C^1 -small, then it can be generated by a C^2 -small time-independent Hamiltonian $H : M \rightarrow \mathbb{R}$
- The fixed points of ϕ are in bijection with the critical points of H .
- It turns out that for $M = T^2$ the number of critical points is at least 3. PLGT encouraged Arnold to make his famous conjecture.
- (Arnold Conjecture, 1965) A general ϕ has at least as many fixed points as the number of critical points of a differentiable function on it. The case where the fixed points of ϕ are assumed to be non-degenerate (generic) corresponds to Morse functions.

The case of surfaces and T^{2n}

- The 2d case was settled by Eliashberg (1979).
- Consider $T^{2n} := (T^2)^n$ with product symplectic form.
- Hamiltonian diffeomorphisms can be directly described through Hamilton's equations, we don't need to refer to symplectic forms.
- Arnold conjecture gives at least $2n + 1$ fixed points (and 2^{2n} for the non-degenerate case).
- Note that LFT gives nothing in this case for arbitrary continuous maps.
- Conley-Zehnder (1983) proved this case.
- Around the same time Gromov non-squeezing was also proved - the first non-trivial results in symplectic geometry.

Morse inequalities and Morse-Witten-Smale cohomology

- In 1934, Morse conjectured that the number of a critical points on a compact smooth manifold Y would be at least the sum of Betti numbers of Y . Proved in 1961 by Smale.
- In 1982, Witten constructed a chain complex generated by the critical points of a Morse function on Y whose homology computes $H^*(Y; \mathbb{Z})$. People say that Smale already knew this in the sixties.
- This immediately proves a strong form of Morse inequalities (including the torsion information in the homology).
- Witten's construction was inspired by supersymmetric quantum mechanics.
- Therefore in the non-degenerate case, Arnold conjecture predicts the sum of Betti numbers many fixed points. Note that LFT predicts the alternating sum!

Floer cohomology of fixed points

- Inspired by Witten's construction and Gromov's J -holomorphic curves, Floer started a research program in late 80's that defines a chain complex generated by the fixed points of a Hamiltonian diffeomorphism $\phi : M \rightarrow M$.
- Under some technical assumptions, he did this and moreover showed that the homology is again isomorphic to the singular homology of M . This settled Arnold Conjecture in a large number of cases.
- After many many years of work, Arnold Conjecture is now considered to be proved in general using Floer's theory.
- Floer unfortunately did not see most of this work because he committed suicide in 1991.
- Floer theory lives forever!

Recommendations for further reading

- For an elementary account of the matters related to the 3-body problem: H. Geiges - The Geometry of Celestial Mechanics
- For a modern perspective on Poincare's transverse sections and related matters: A. Moreno - The Symplectic Geometry of the Three-Body Problem
- For a streamlined account of Conley-Zehnder's proof: D. McDuff, D. Salamon - Introduction to Symplectic Topology
- For a quick and terse introduction to Floer theory: D. Salamon - Lectures on Floer homology
- For a detailed account of Morse and Floer theory: M. Audin, M. Damian - Morse Theory and Floer Homology