



Symplectic/contact displaceability in low dimensions and big fiber theorems

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Outline

- 1 Displacement on the sphere and its product
- 2 Standard contact sphere and the Hopf map
- 3 Givental's Theorem
- 4 Big Fiber Theorems

Isotopies of the sphere

- Consider the unit sphere $S^2 \subset \mathbb{R}^3$ with its standard area form ω and total area $A = 4\pi$.
- An isotopy is a smooth family of diffeomorphisms $\{\phi_t\}_{t \in [0,1]}$ of S^2 with $\phi_0 = \text{id}$.
- It is **area preserving** if $\phi_t^* \omega = \omega$ for every t .
- Let $K \subset S^2$ be nonempty, closed and connected. We say that the isotopy **displaces K from itself** if

$$\phi_1(K) \cap K = \emptyset.$$

Displacement by an isotopy

- **Question 1:** When can K be displaced from itself by an isotopy?
- **Answer:** If and only if $K \neq S^2$.
- Choose the north pole outside K . The downward vertical vector field, projected onto TS^2 , has a flow squeezing K into an arbitrarily small disk near the south pole.
- A rotation then moves this disk into $S^2 \setminus K$.

Displacement by an area preserving isotopy

- **Question 2:** When can K be displaced from itself by an area preserving isotopy?
- **Answer:** If and only if $S^2 \setminus K$ contains a smoothly embedded closed disk D with

$$\text{area}(D) = \frac{1}{2} \text{area}(S^2).$$

- This is a restriction on the areas of the complementary regions of K .

Why the half-area disk is sufficient

- Suppose that a closed disk $D \subset S^2 \setminus K$ has area $A/2$.
- By Moser's argument, there is an area preserving diffeomorphism $f : S^2 \rightarrow S^2$ taking D to a closed hemisphere.
- Then $f(K)$ lies in the opposite open hemisphere. A half-turn R exchanges the two open hemispheres, so

$$R(f(K)) \cap f(K) = \emptyset.$$

- Conjugating the rotation isotopy by f gives an area preserving isotopy displacing K .

Moser's argument

- Smooth Schoenflies gives an orientation preserving $h : S^2 \rightarrow S^2$ taking D to a closed hemisphere: glue parametrizations of the two disks that agree in boundary collars. Set $C = \partial D$ and $\omega_1 = h^*\omega$.
- Equal total areas give $\omega_1 - \omega = d\alpha$. The disk areas give $\int_C \alpha = \int_D (\omega_1 - \omega) = 0$. Thus $\alpha|_C = dg$. Extend g to $\tilde{g}$ on S^2 and put $\beta = \alpha - d\tilde{g}$, so $\beta|_{TC} = 0$.
- Set $\omega_t = (1 - t)\omega + t\omega_1$ and $\iota_{X_t}\omega_t = -\beta$ on all of S^2 . Compactness gives a global flow ψ_t for $0 \leq t \leq 1$. Tangency to C ensures $\psi_t(D) = D$.
- The identity $\frac{d}{dt}(\psi_t^*\omega_t) = \psi_t^*(d\beta + d\iota_{X_t}\omega_t) = 0$ gives $\psi_1^*\omega_1 = \omega$. Hence $f = h \circ \psi_1 : S^2 \rightarrow S^2$ is the required global area preserving diffeomorphism.

Why the half-area disk is necessary

- Suppose that an area preserving ϕ displaces K . Choose a smooth compact connected **domain** N (a codimension 0 submanifold with boundary) with $K \subset \text{int } N$ and $\phi(N) \cap N = \emptyset$.
- The components of $S^2 \setminus N$ are open disks. The connected set $\phi(N)$ lies in one of them, say D .
- Since $\phi(\partial D) \subset D$, the Jordan curve theorem gives either $\phi(\overline{D}) \subset D$ or $\phi(S^2 \setminus D) \subset D$. Area preservation rules out the first.
- Thus $A - \text{area}(D) < \text{area}(D)$, so $\text{area}(D) > A/2$. Inside D we can find a smooth closed disk of area $A/2$.

Symplectic area in $S^2 \times S^2$

- On S^2 , the signed area of a parametrized oriented surface $u : \Sigma \rightarrow S^2$ is $\int_{\Sigma} u^* \omega$.
- On $S^2 \times S^2$, consider the symplectic form

$$\Omega = \text{pr}_1^* \omega + \text{pr}_2^* \omega.$$

This is a closed, non-degenerate differential 2-form.

- The **symplectic area** of $u = (u_1, u_2) : \Sigma \rightarrow S^2 \times S^2$ is

$$\int_{\Sigma} u^* \Omega = \int_{\Sigma} u_1^* \omega + \int_{\Sigma} u_2^* \omega.$$

Hamiltonian isotopies

- An isotopy $\{\phi_t\}$ of $S^2 \times S^2$ is **symplectic** if $\phi_t^* \Omega = \Omega$ for every t .
- It is **Hamiltonian** if it is the flow of a time dependent vector field X_{H_t} satisfying

$$\iota_{X_{H_t}} \Omega = -dH_t$$

for smooth functions $H_t : S^2 \times S^2 \rightarrow \mathbb{R}$.

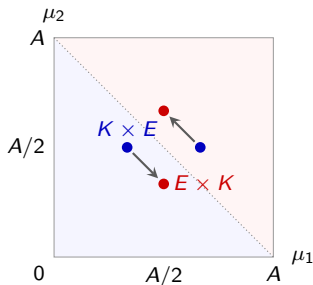
- Hamiltonian isotopies preserve Ω . Conversely, every symplectic isotopy here is Hamiltonian since $H^1(S^2 \times S^2; \mathbb{R}) = 0$.
- On S^2 , Hamiltonian isotopies are precisely the area preserving isotopies, since $H^1(S^2; \mathbb{R}) = 0$.

A product theorem in dimension 4

- **Theorem:** Let $K_1, K_2 \subset S^2$ be closed and connected. Then $K_1 \times K_2 \subset S^2 \times S^2$ is Hamiltonian displaceable if and only if at least one of K_1, K_2 is area preserving displaceable in S^2 .
- One direction is easy: a displacing isotopy in one factor gives a product Hamiltonian isotopy.
- The other direction is hard! A general Hamiltonian isotopy of $S^2 \times S^2$ need not respect the product structure.

Connectedness is essential

- Let $K = C_- \sqcup C_+$ be two parallels dividing S^2 into three regions of area $A/3$, and let E be the equator.
- Both K and E are area preserving **nondisplaceable**, by area considerations.



The two blue points represent $K \times E$ in the moment square $[0, A]^2$.

The regions below and above $\mu_1 + \mu_2 = A$ lift to disjoint standard symplectic balls.

Hamiltonian coordinate swaps, supported in these balls, take the blue fibers to the red fibers.

$$\phi(K \times E) = E \times K, \quad (E \times K) \cap (K \times E) = \emptyset.$$

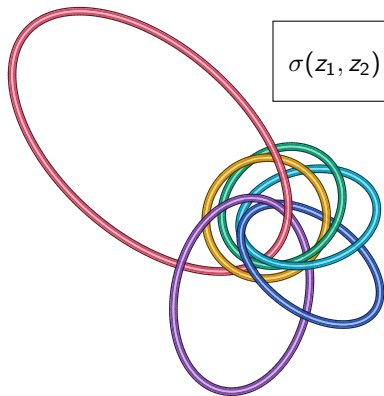
Hopf fibration

- From now on, $S^3 \subset \mathbb{C}^2$ has radius 1 and $S^2 \subset \mathbb{R}^3$ has radius $1/\sqrt{2}$, hence total area 2π .
- The Hopf fibration $q : S^3 \rightarrow S^2$ is

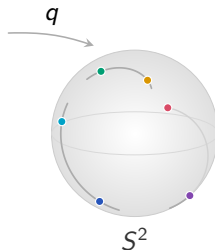
$$(z_1, z_2) \mapsto \left(\sqrt{2} \bar{z}_1 z_2, \frac{|z_1|^2 - |z_2|^2}{\sqrt{2}} \right).$$

- Its fibers are intersections of S^3 with complex lines through 0.
- The diagonal action $\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$ makes q a principal $\mathbb{R}/2\pi\mathbb{Z}$ -bundle.
- Writing $z_j = x_j + iy_j$, the connection form $\alpha = \sum_j (x_j dy_j - y_j dx_j)$ satisfies $\ker \alpha = TS^3 \cap iTS^3 = \xi$ and $d\alpha = q^* \omega_{\text{area}}$.

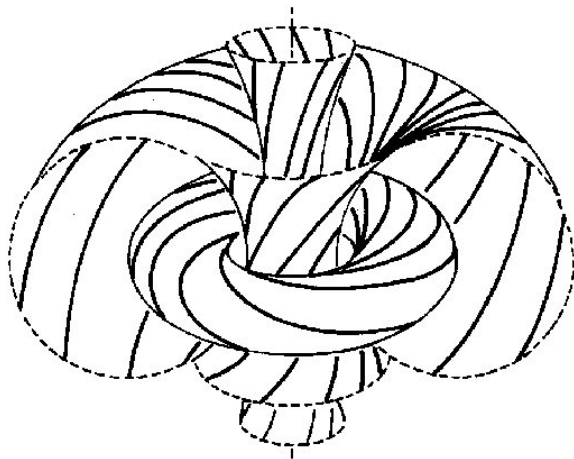
Hopf fibres after stereographic projection from $(0, i)$



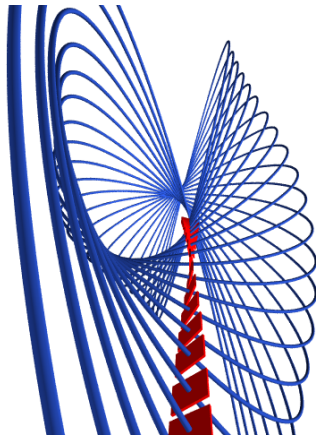
$$\sigma(z_1, z_2) = \frac{(\operatorname{Re} z_1, \operatorname{Im} z_1, \operatorname{Re} z_2)}{1 - \operatorname{Im} z_2} \in \mathbb{R}^3.$$



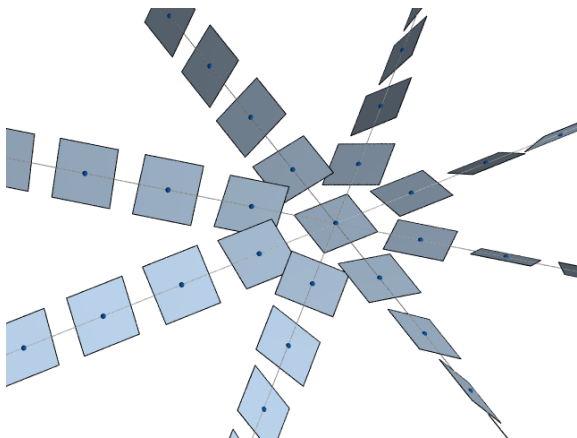
Hopf fibration after stereographic projection



Contact planes in relation to Hopf fibers



Contact planes altogether

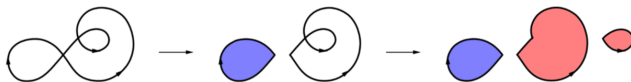


Contact geometry

- The plane field $\xi = \ker \alpha$ is a contact structure: $\alpha \wedge d\alpha$ is nowhere zero.
- The maximal dimension of a submanifold tangent to ξ is 1. Such curves are called **Legendrian**; they may be embedded or immersed.
- The **Reeb field** satisfies $\alpha(R_\alpha) = 1$ and $\iota_{R_\alpha} d\alpha = 0$. Here its flow is the Hopf circle action.
- A **contact isotopy** is a smooth family ϕ_t with $\phi_0 = \text{id}$ and $\phi_t^* \alpha = c_t \alpha$, where the positive function c_t is the **conformal factor**.
- Hamiltonian isotopies of S^2 lift to contact isotopies of S^3 .
- An isotopy of **embedded Legendrian loops** extends to an ambient contact isotopy.

Legendrians as immersed paths in the base S^2

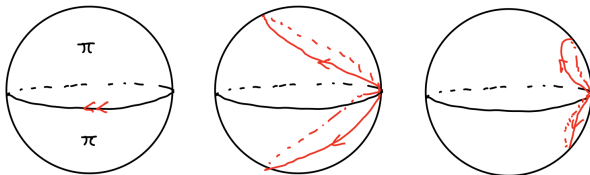
- Any immersed Legendrian path in S^3 projects to an immersed path in S^2 .
- Conversely, an immersed path in S^2 has a unique Legendrian lift once its initial point is chosen.
- (Bohr-Sommerfeld condition) An immersed loop downstairs closes up upstairs iff the signed area of a capping disk is an integer multiple of 2π .



- A closed lift is embedded iff no proper subloop satisfies this condition.

Full squeezing in standard contact S^3

- Any $K \subset S^3$ that is not dense can be squeezed into any nonempty open subset of S^3 by a contact isotopy.
- In particular, any embedded Legendrian loop can be moved into an arbitrary neighborhood of a point by a contact isotopy.
- We can explicitly do this for a Legendrian lift of the twice wrapped equator:



The statement of Givental's theorem

- Quotient by the antipodal $\mathbb{Z}_2$ -action ($x \sim -x$) on S^3 gives rise to

$$(\mathbb{RP}^3, \xi_2, \alpha_2),$$

with induced map $q_2 : \mathbb{RP}^3 \rightarrow S^2$.

- This is a principal $\mathbb{R}/\pi\mathbb{Z}$ -bundle. The Bohr–Sommerfeld condition is now signed area in $\pi\mathbb{Z}$.
- A hemisphere has area π , so an equator E lifts to a closed Legendrian loop L . Choose one such lift.
- **Theorem** (Givental, 1990): L cannot be disjointed from $q_2^{-1}(E)$ by a contact isotopy!

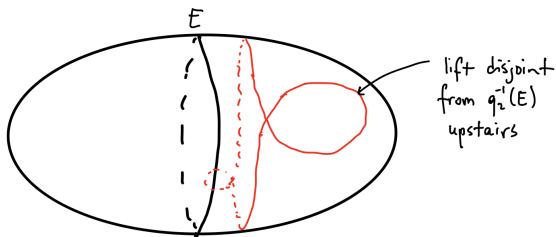
Whitney move

- This can be reinterpreted entirely as a statement about immersed loops in S^2 .
- The following Whitney move is a local deformation through immersed curves:



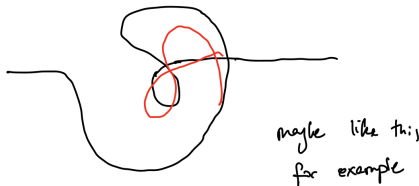
What we can achieve with the Whitney move

- Using Whitney's move, modified to preserve the Bohr–Sommerfeld condition, we can deform the equator to the pictured loop through immersed Bohr–Sommerfeld loops:



The conclusion of Givental's result

- Any such deformation through immersed Bohr–Sommerfeld loops must at some point produce a lift that **ceases to be embedded**. Equivalently, a proper subloop has signed area in $\pi\mathbb{Z}$, so two distinct points of the lift coincide.



Final remarks on Givental's result

- Givental also proved the analogous result for the standard prequantization $\mathbb{RP}^{2n+1} \rightarrow \mathbb{CP}^n$ and the Clifford torus, the monotone Lagrangian fiber of the toric moment map.
- Borman–Zapolsky (2015) and GKPS (2021) generalized this to other toric cases. In joint work with Sun and Uljarevic (2025), we gave a generalization using symplectic cohomology.
- Even for the $n = 1$ case no simple proof of Givental's Theorem is known. I will continue with this case in focus for a bit more.

Translation to big fiber theorem

- Consider the map

$$p : (\mathbb{RP}^3, \xi_2) \rightarrow \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$$

obtained by composing q_2 with the height map.

- Its value is constant along the Reeb circles. A single Reeb invariant function is a contact involutive map, witnessed here by α_2 .
- Our result says that, if some condition holds, then we can find a "big" fiber, where "big" means non-displaceable from itself by a contact isotopy.
- In fact, we guarantee an intersection point (with its image) with α_2 -conformal factor 1. This implies Givental's result.

Symplectic big fiber theorem

- Our inspiration is the symplectic big fiber theorem of Entov-Polterovich.
- **Theorem** (EP, 2006): Let (M, ω) be a closed symplectic manifold and $\pi : M \rightarrow \mathbb{R}^N$ a smooth map with Poisson commuting components. Then, π has at least one fiber that is not Hamiltonian displaceable from itself.
- Poisson commuting means $\{\pi_i, \pi_j\} = \omega(X_{\pi_i}, X_{\pi_j}) = 0$ for all i, j .
- I gave an alternative proof using a Mayer-Vietoris property for symplectic cohomology with support (2021).

Contact big fiber theorem: Sun-Uljarevic-V. (2025)

- Let Y be closed with contact form α , and let $\pi : Y \rightarrow \mathbb{R}^N$ be smooth with Reeb invariant, Jacobi commuting components.
- The S^3 example shows that extra hypotheses are needed. Either criterion below suffices. Both hold for $(\mathbb{R}P^3, \xi_2)$.
- There is a Liouville filling with nonzero symplectic cohomology. Here we can take the disk cotangent bundle D^*S^2 , whose completion is T^*S^2 .
- Y is a prequantization of a positively monotone closed symplectic manifold with minimal Chern number at least 2, and its Euler class is not invertible in quantum cohomology over some coefficient ring. Here the Euler class vanishes modulo 2.

The geometric argument for Givental's theorem

- Lifted rotations displace every nonzero fiber of p . Hence $T = q_2^{-1}(E)$ must have an intersection point with conformal factor 1 under every contact isotopy.

- Suppose $\phi(L) \cap T = \emptyset$. Write

$$\phi^* \alpha_2 = c \alpha_2, \quad m = \min c > 0.$$

- Near T , use coordinates with

$$\alpha_2 = d\theta + s dx, \quad T = \{s = 0\}, \quad L = \{s = \theta = 0\},$$

where $\theta \in \mathbb{R}/\pi\mathbb{Z}$ is the fiber coordinate.

- Choose a band $U = S_x^1 \times I_\theta \subset T$ around L small enough that $\phi(U) \cap T = \emptyset$.

The geometric argument: the contradiction

- Choose an orientation preserving diffeomorphism f of the θ -circle with $(f^{-1})' < m$ outside I_θ .

- Near T , set

$$\psi(x, s, \theta) = (x, f'(\theta)s, f(\theta)), \quad \psi^*\alpha_2 = f'(\theta)\alpha_2.$$

Choose an isotopy f_t and cut off its generating contact Hamiltonian, with cutoff 1 near T , to extend globally.

- If $z, \phi\psi(z) \in T$, then $\psi(z) \notin U$, so $f(\theta) \notin I_\theta$ and

$$f'(\theta) = \frac{1}{(f^{-1})'(f(\theta))} > \frac{1}{m}.$$

- Thus the conformal factor of $\phi \circ \psi$ at every such intersection is $c(\psi(z))f'(\theta) > 1$, contradicting the big fiber theorem.

Exercise: the product theorem in dimension 4

- **Exercise.** Use the Entov–Polterovich big fiber theorem to prove: for closed connected $K_1, K_2 \subset S^2$, if $K_1 \times K_2$ is Hamiltonian displaceable, then at least one K_i is area preserving displaceable.
- Prove the contrapositive, first for $K_i = N_i$, where N_i are smooth compact connected **domains**.
- Assume both N_i are non-displaceable. Put $A = \text{area}(S^2)$. Why are the components U_{ij} of $S^2 \setminus N_i$ finitely many open disks of area $\leq A/2$?
- We can choose smooth $h_{ij} \geq 0$, positive exactly on U_{ij} , one per disk. Put $F_i = (h_{i1}, \dots, h_{ir_i})$. Their positive sets are disjoint and $dh_{ij} = 0$ wherever $h_{ij} = 0$, so the components Poisson commute. Show $F_i^{-1}(0) = N_i$.

Hint: prove the contrapositive

- A nonzero fiber of F_i is compact in one U_{ij} . Enclose it in a smooth disk of area $< A/2$, even if $\text{area}(U_{ij}) = A/2$. Thus every fiber except $F_i^{-1}(0) = N_i$ is Hamiltonian displaceable.
- Set $\Phi(x, y) = (F_1(x), F_2(y))$. Its components commute. If $(a, b) \neq (0, 0)$, one factor of $F_1^{-1}(a) \times F_2^{-1}(b)$ is displaceable, so this product fiber is displaceable.
- The big fiber theorem therefore forces $\Phi^{-1}(0, 0) = N_1 \times N_2$ to be non-displaceable.
- **General case:** for non-displaceable K_i , choose small domain neighborhoods $K_i \subset \text{int } N_i$. They remain non-displaceable since they contain K_i . By compactness, a displacement of $K_1 \times K_2$ would also displace $N_1 \times N_2$ for sufficiently small N_i .

Entov–Polterovich,

Quasi-states and symplectic intersections, Corollary 2.5.

Contact big fiber theorem (Sun–Uljarevic–V.)

- **Theorem.** On a closed contact manifold (Y, α) , let $\pi : Y \rightarrow \mathbb{R}^N$ be smooth with Reeb invariant, pairwise Jacobi commuting components. Assume either:
- **Liouville criterion:** $(Y, \ker \alpha)$ has a Liouville filling W with $SH^*(W; \mathbb{k}) \neq 0$ for some commutative ring $\mathbb{k}$.
- **Prequantization criterion:** $q : (Y, \alpha) \rightarrow (B, \omega)$ is a prequantization over a closed positively monotone base with minimal Chern number ≥ 2 . For some commutative ring $\mathbb{k}$, quantum multiplication by the Euler class $e(q)$ is not invertible on $H^*(B; \mathbb{k})[T, T^{-1}]$.
- **Conclusion:** some fiber K satisfies: every ϕ contact isotopic to the identity has $x \in K$ with

$$\phi(x) \in K, \quad (\phi^* \alpha)_x = \alpha_x.$$

Proofs and references

- If there is time, we will discuss the proofs of the big fiber theorems on the blackboard.
- U. Varolgüneş, *Mayer–Vietoris property for relative symplectic cohomology*. *Geometry & Topology* **25** (2021), 547–642. [arXiv:1806.00684](#).
- U. Varolgüneş, *Mayer–Vietoris property for relative symplectic cohomology*. Ph.D. thesis, MIT, 2018.
- Y. Sun, I. Uljarević, and U. Varolgunes, *Contact big fiber theorems*, *Geom. Funct. Anal.* **36** (2026), 301–350.

Appendix: Poisson and Jacobi commuting

- On a symplectic manifold (M, ω) , define

$$\iota_{X_f}\omega = -df, \quad \{f, g\} = \omega(X_f, X_g) = X_f(g).$$

A family is **Poisson commuting** if $\{f_i, f_j\} = 0$ for all i, j .

- On (Y, α) with Reeb field R_α , the **contact Hamiltonian vector field** X_f is defined by

$$\alpha(X_f) = f, \quad \iota_{X_f}d\alpha = (R_\alpha f)\alpha - df.$$

The **Jacobi bracket** is

$$[f, g]_\alpha = \alpha([X_f, X_g]) = X_f(g) - gR_\alpha f.$$

A family is **Jacobi commuting** if $[f_i, f_j]_\alpha = 0$ for all i, j .

- Our contact big fiber theorem also requires $R_\alpha f_i = 0$. For a prequantization $d\alpha = q^*\omega$,

$$[q^*F, q^*G]_\alpha = q^*\{F, G\}.$$